Kinetic theory of point vortices: Diffusion coefficient and systematic drift

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We develop a kinetic theory for point vortices in two-dimensional hydrodynamics. Using standard projection operator techniques, we derive a Fokker-Planck equation describing the relaxation of a "test" vortex in a bath of "field" vortices at statistical equilibrium. The relaxation is due to the combined effect of a diffusion and a drift. The drift is shown to be responsible for the organization of point vortices at negative temperatures. A description that goes beyond the thermal bath approximation is attempted. A new kinetic equation is obtained which respects all conservation laws of the point vortex system and satisfies a H theorem. Close to equilibrium, this equation reduces to the ordinary Fokker-Planck equation.

I. INTRODUCTION

It is often useful in two-dimensional turbulence to approximate a continuous vorticity field by a cloud of point vortices. The main interest is that such a system is Hamiltonian [1] and can be studied by rather ordinary statistical mechanics. This was first considered by Onsager [2], who showed qualitatively the existence of negative temperature states at which vortices cluster. He could therefore explain the occurrence of large-scale vortices (or "supervortices") often observed in nature. This was a remarkable anticipation since observations were very scarce at that time. His work was pursued by Joyce and Montgomery [3] and Lundgren and Pointin [4], who introduced a mean-field approximation and obtained explicit results for the equilibrium state. They derived in particular a Maxwell-Boltzmann statistics for the distribution of point vortices at equilibrium.

Less is known concerning the relaxation towards equilibrium. In fact, the evolution of the N-particle distribution function is governed by a Liouville equation, but this equation contains too much information to be of practical use. One is more interested in the evolution of the one-particle distribution function $P(\mathbf{r}, t)$, which gives the probability that a point vortex will be found in $\mathbf{r}$ at time t. In Ref. [5], we have described the relaxation of $P(\mathbf{r}, t)$ towards the Boltzmann distribution in terms of a phenomenological Fokker-Planck equation. In this approach, the vortices have a diffusive motion due to random fluctuations and they experience in addition a *systematic drift* [5] directed along the background density gradient. Physically, the drift is the result of a polarization process and its mathematical expression can be determined with a linear-response theory [5]. It is found that the drift is "attractive" at negative temperatures so the vortices cluster into macrovortices in agreement with Onsager's thermodynamical approach. At equilibrium, the drift balances the scattering and maintains a nontrivial vortex distribution (the Boltzmann distribution) providing a dynamical explanation for the persistence of clustering.

In this paper, we justify our phenomenological model by deriving the Fokker-Planck equation directly from the Liouville equation, using projection operator techniques [6]. These methods are standard in statistical mechanics but they

are applied here to a system of vortices. We first consider the relaxation of a "test" vortex in a thermal bath in which the "field" vortices are in statistical equilibrium. In this approximation, the Fokker-Planck equation appears in its usual form with a diffusion term and a drift term. The drift coefficient is connected to the diffusion coefficient and to the temperature of the bath $1/\beta$ by an Einstein formula. The diffusion coefficient is expressed as a Kubo formula, i.e., as the integral of the velocity autocorrelation function. Using an approximation in which the vortices are advected by the equilibrium flow, we find that the autocorrelation function decays like t^{-2} for large times [5]. This is a slow decay but it ensures the convergence of the diffusion coefficient. We also derive non-Markovian equations that keep track of memory effects. Then, we relax the thermal bath approximation and derive a generalized kinetic equation for our vortex system. This integrodifferential equation satisfies all conservation laws of the point vortex system and increases the Boltzmann entropy (H theorem). The relaxation towards equilibrium is due to a condition of resonance. If this condition is not satisfied, the system can remain frozen in a sort of "metastable" equilibrium. In contrast, if the system is sufficiently resonant, it will converge towards the maximum entropy state described by the Boltzmann distribution. Close to equilibrium, our generalized kinetic equation reduces to the ordinary Fokker-Planck equation.

The methods developed in this paper are inspired by those introduced in plasma physics and stellar dynamics [7]. In particular, the *systematic drift* [5] of a point vortex is the counterpart of the *dynamical friction* [8] experienced by a star in a stellar system. Further analogies between two-dimensional (2D) vortices and stellar systems are discussed in the paper and in [9–14]. Other kinetic theories of point vortices have been developed in Ref. [34] in a different context. A good review on point vortex dynamics is given by Newton [35].

II. STATISTICAL MECHANICS OF POINT VORTICES

A. The point vortex model

In a two-dimensional incompressible fluid, the velocity field $\mathbf{u}$ is divergenceless and can be written in terms of a stream function ψ in the form

$$\mathbf{u} = -\mathbf{z} \times \nabla \psi, \tag{1}$$

where $\mathbf{z}$ is a unit vector normal to the flow. The stream function is related to the vorticity $\omega \mathbf{z} = \nabla \times \mathbf{u}$ by the Poisson equation

$$\omega = -\Delta \psi \tag{2}$$

obtained by taking the curl of Eq. (1). The impermeability condition implies that ψ is constant on the boundary and we shall take $\psi = 0$ by convention.

We shall consider the situation in which the velocity is created by a collection of N point vortices of equal circulation γ. In that case, the vorticity field can be expressed as a sum of δ functions in the form

$$\omega(\mathbf{r}, t) = \sum_{i=1}^{N} \gamma \delta(\mathbf{r} - \mathbf{r}_i(t)), \tag{3}$$

where $\mathbf{r}_i(t)$ denotes the position of point vortex i at time t. Its velocity is given by

$$\mathbf{V}_i = \frac{d\mathbf{r}_i}{dt} = -\mathbf{z} \times \nabla \psi(\mathbf{r} = \mathbf{r}_i, t), \tag{4}$$

where ψ is a solution of the Poisson equation (2) with the vorticity field (3). In an unbounded fluid, one has

$$\psi(\mathbf{r}) = -\frac{1}{2\pi} \sum_{i=1}^{N} \gamma \ln|\mathbf{r} - \mathbf{r}_i|. \tag{5}$$

Therefore, the velocity of a point vortex is equal to the sum of the velocities $\mathbf{V}(j \rightarrow i)$ produced by the $N-1$ other vortices, i.e.,

$$\mathbf{V}_i = \sum_{j \neq i} \mathbf{V}(j \rightarrow i) \tag{6}$$

with

$$\mathbf{V}(j \rightarrow i) = -\frac{\gamma}{2\pi} \mathbf{z} \times \frac{\mathbf{r}_j - \mathbf{r}_i}{|\mathbf{r}_j - \mathbf{r}_i|^2}. \tag{7}$$

The above dynamics can be cast in a Hamiltonian form [1]:

$$\gamma \frac{dx_i}{dt} = \frac{\partial H}{\partial y_i}, \quad \gamma \frac{dy_i}{dt} = -\frac{\partial H}{\partial x_i}, \tag{8}$$

$$H = -\frac{1}{4\pi} \sum_{i \neq j} \gamma^2 \ln|\mathbf{r}_i - \mathbf{r}_j|, \tag{9}$$

where the coordinates (x, y) of the point vortices are canonically conjugate. These equations of motion still apply when the fluid is restrained by boundaries, in which case the Hamiltonian (9) is modified so as to allow for vortex images, and may be constructed in terms of Green's functions depending on the geometry of the domain. Since H is not explicitly time dependent, it is a constant of the motion and it represents the "potential" energy of the point vortices (we

shall see later on that it also represents the kinetic energy of the flow). Therefore, point vortices behave like particles in interactionlike electric charges or stars. Note, however, that the Hamiltonian (9) does not involve a "kinetic" energy of the vortices in the usual sense. This is related to the particular circumstance that a point vortex produces a velocity, not an acceleration. As a result, an isolated vortex remains at rest, contrary to a material particle, which has a rectilinear motion due to its inertia. Point vortices form, therefore, a very peculiar Hamiltonian system.

B. The microcanonical approach of Onsager

The statistical mechanics of point vortices was first considered by Onsager [2], who showed the existence of negative temperatures at which point vortices cluster into "supervortices." Let us briefly recall his argumentation.

Consider a liquid enclosed by a boundary, so that the vortices are confined to an area A. Since the coordinates (x, y) of the point vortices are canonically conjugate, the phase space coincides with the configuration space and is *finite*:

$$\int dx_1 dy_1 \cdots dx_N dy_N = \left(\int dx \, dy \right)^N = A^N. \tag{10}$$

This striking property contrasts with most classical Hamiltonian systems considered in statistical mechanics, which have unbounded phase spaces due to the presence of a kinetic term in the Hamiltonian.

As is usual in the microcanonical description of a system of N particles, we introduce the density of states,

$$g(E) = \int dx_1 dy_1 \cdots dx_N dy_N \delta(E - H(x_1, y_1, \ldots, x_N, y_N)), \tag{11}$$

which gives the phase-space volume per unit interaction energy E. The phase-space volume, which corresponds to energies H less than a given value E, can be written

$$\Phi(E) = \int_{E_{\min}}^{E} g(E) dE. \tag{12}$$

It increases monotonically from zero to A^N when E goes from $E_{\min}$ to $+\infty$. Therefore, $g(E) = d\Phi(E)/dE$ will have a maximum value at some $E = E_m$, say, before decreasing to zero when $E \rightarrow +\infty$.

In the microcanonical ensemble, the entropy and the temperature are defined by

$$S = \ln g(E) \quad \beta = \frac{1}{T} = \frac{dS}{dE}. \tag{13}$$

For $E > E_m$, $S(E)$ is a decreasing function of energy and consequently the temperature is *negative*. Now, high-energy states $E \gg E_m$ are clearly those in which the vortices are crowded as close together as possible. For energies only slighlty greater than E_m, the concentration will not be so dramatic but there will be a tendency for the vortices to group themselves together on a macroscopic scale and form

"clusters" or "supervortices." In contrast, for $E < E_m$, the temperature is positive and the vortices have the tendency to accumulate on the boundary of the domain in order to decrease their energy. For a system with positive and negative vortices, the negative temperature states, achieved for relatively high energies, consist of two large counter-rotating vortices physically well separated in the box. On the contrary, when $E \to -\infty$, the temperature is positive and vortices of opposite circulation tend to pair off.

C. The mean-field approximation

It is easy to show that the exact distribution of point vortices expressed in terms of δ functions,

$$\omega_{\text{exact}}(\mathbf{r},t) = \sum_{i=1}^{N} \gamma \delta(\mathbf{r} - \mathbf{r}_i(t)), \tag{14}$$

is a solution of the Euler equation,

$$\frac{\partial \omega_{\text{ex}}}{\partial t} + \mathbf{u}_{\text{ex}} \boldsymbol{\nabla} \omega_{\text{ex}} = 0, \tag{15}$$

where $\mathbf{u}_{\text{ex}}$ is the exact velocity field determined by Eqs. (1), (2), and (14). This is proved as follows. Taking the derivative of Eq. (14) with respect to time, we obtain

$$\frac{\partial \omega_{\text{ex}}}{\partial t} = -\sum_{i=1}^{N} \gamma \boldsymbol{\nabla} \delta(\mathbf{r} - \mathbf{r}_i(t)) \mathbf{V}_i. \tag{16}$$

Using $\mathbf{V}_i = \mathbf{u}_{\text{ex}}(\mathbf{r}_i(t),t)$, we can rewrite the foregoing equation in the form

$$\frac{\partial \omega_{\text{ex}}}{\partial t} = -\boldsymbol{\nabla} \sum_{i=1}^{N} \gamma \delta(\mathbf{r} - \mathbf{r}_i(t)) \mathbf{u}_{\text{ex}}(\mathbf{r},t). \tag{17}$$

Since the velocity is divergenceless, we obtain

$$\frac{\partial \omega_{\text{ex}}}{\partial t}(\mathbf{r},t) = -\mathbf{u}_{\text{ex}}(\mathbf{r},t) \boldsymbol{\nabla} \sum_{i=1}^{N} \gamma \delta(\mathbf{r} - \mathbf{r}_i(t)) = -\mathbf{u}_{\text{ex}} \boldsymbol{\nabla} \omega_{\text{ex}}(\mathbf{r},t). \tag{18}$$

Therefore, the Euler equation (15) with Eqs. (1), (2), and (3) contains exactly the same information as the Hamiltonian system (8) and (9).

This description in terms of δ functions, while being technically correct, is useless for practical purposes, because it requires the knowledge of the exact trajectories of the point vortices for an arbitrary initial condition [or the solution of the Euler equation (15)]. When N is large, this task is impossibly difficult. Therefore, instead of the exact vorticity field expressed in terms of δ functions, one is more interested in functions that are smooth. For that reason, we introduce a smooth vorticity field $\langle \omega \rangle(\mathbf{r},t)$, which is proportional to the average number of vortices contained in the cell $(\mathbf{r},\mathbf{r}+d\mathbf{r})$ at time t. This mean-field description, which ignores the granularities of the sytem, requires that it be possible to divide the domain in a large number of cells in such a way that each cell is (a) large enough to contain a macroscopic number of

point vortices, but (b) small enough for all the particles in the cell to be assumed to possess the same average characteristic of the cell.

In this mean-field approximation, the Hamiltonian (9) is changed into

$$E = -\frac{1}{4\pi} \int \langle \omega \rangle(\mathbf{r}) \langle \omega \rangle(\mathbf{r}') \ln|\mathbf{r} - \mathbf{r}'| d^2\mathbf{r} d^2\mathbf{r}'. \tag{19}$$

In writing this expression, we have not taken into account the constraint $j \neq i$ appearing in Eq. (9). Really, in Eq. (19) the integration extends over the point $\mathbf{r} = \mathbf{r}'$ so that Eq. (19) contains self-energy terms that become infinitely large for point vortices. As will soon become apparent, this mean-field approximation implies that the energy E is positive, a property that is not necessarily shared by the Hamiltonian (9).

Using Eq. (5), adequately generalized to account for a continuous distribution of vortices, our expression (19) for E can be rewritten

$$E = \frac{1}{2} \int \langle \omega \rangle \psi d^2\mathbf{r}. \tag{20}$$

Introducing explicitly the Poisson equation in Eq. (20) and integrating by parts, one has successively

$$E = \frac{1}{2} \int \psi(-\Delta \psi) d^2\mathbf{r} = \frac{1}{2} \int (\boldsymbol{\nabla}\psi)^2 d^2\mathbf{r} = \int \frac{\langle \mathbf{u} \rangle^2}{2} d^2\mathbf{r}, \tag{21}$$

where $\langle \mathbf{u} \rangle$ is the smooth velocity field (the second equality is obtained by a part integration with the condition $\psi = 0$ on the boundary). Therefore, E can be interpreted either as the potential energy of interaction between vortices [see Eq. (20)] or as the kinetic energy of the flow [see Eq. (21)].

D. The mean-field equilibrium

We now wish to determine the distribution of vortices at equilibrium following a statistical mechanics approach [3]. To that purpose, using the Boltzmann procedure, we divide the macrocells $(\mathbf{r},\mathbf{r}+d\mathbf{r})$ into a large number of microcells and enumerate the number of "microstates" that correspond to the same "macroscopic" configuration of the system. The logarithm of this number defines the entropy. In the mean-field approximation, this leads to the classical formula

$$S = -N \int P(\mathbf{r}) \ln P(\mathbf{r}) d^2\mathbf{r}, \tag{22}$$

where $P(\mathbf{r})$ is the density probability that a point vortex will be in the surface element centered on $\mathbf{r}$. The average vorticity in $\mathbf{r}$ is related to this probability density by

$$\langle \omega \rangle(\mathbf{r}) = N \gamma P(\mathbf{r}). \tag{23}$$

At equilibrium, the system is in the most probable macroscopic state, i.e., the state that is the most represented at the microscopic level. This optimal state is obtained by maximizing the Boltzmann entropy (22) at fixed energy (20) and vortex number N, or total circulation

$$\Gamma = N\gamma = \int \langle \omega \rangle d^2\mathbf{r}. \tag{24}$$

Writing the variational principle in the form

$$\delta S - \beta \delta E - \alpha \delta \Gamma = 0, \tag{25}$$

where β and α are Lagrange multipliers, we find that the maximum entropy state corresponds to the Boltzmann distribution [3],

$$\langle \omega \rangle = A e^{-\beta\gamma\psi}, \tag{26}$$

with inverse temperature β. We can account for the conservation of angular momentum $L = \int \langle \omega \rangle r^2 d^2\mathbf{r}$ (in a circular domain) and impulse $P = \int \langle \omega \rangle y d^2\mathbf{r}$ (in a channel) by introducing appropriate Lagrange multipliers Ω and U for each of these constraints. In that case, Eq. (26) remains valid provided that we replace the stream function ψ by the relative stream function $\psi' = \psi + (\Omega/2)r^2 - Uy$. This more general situation has been considered in, e.g., Ref. [15] to describe rotating or translating dipoles.

Substituting the Boltzmann relation between $\langle \omega \rangle$ and ψ in the Poisson equation (2), we obtain a differential equation for the stream function:

$$-\Delta\psi = A e^{-\beta\gamma\psi} \quad \text{(point vortices)}, \tag{27}$$

which determines the equilibrium distribution of vortices. In the case of stellar systems and electric charges, the corresponding Boltzmann-Poisson equation has the form [16]

$$\Delta\Phi = 4\pi G A e^{-\beta' m\Phi} \quad \text{(stellar systems)}, \tag{28}$$

$$-\Delta\Phi = \frac{A}{\epsilon_0} e^{-\beta' q\Phi} \quad \text{(electric charges)}, \tag{29}$$

where Φ denotes successively the gravitational and the electrostatic potential. For these systems, $\beta' > 0$ since the temperature is a measure of the kinetic energy. In contrast, for point vortices there is no kinetic term in the Hamiltonian (9) and the temperature can be either positive or negative. When $\beta < 0$, Eq. (27) is similar in structure to Eq. (28). The vortices tend to attract each other, like stars in a galaxy, and form "clusters" or "supervortices." The density profile determined by Eq. (27) or Eq. (28) is a *decreasing* function of the distance. When $\beta > 0$, Eq. (27) is similar in structure to Eq. (29). The vortices tend to repel each other, like electric charges, and accumulate at the boundary. The density profile determined by Eq. (27) or Eq. (29) is an *increasing* function of the distance. Therefore, the formal analogy between 2D vortices and stellar systems is intimately related to the existence of negative temperatures in 2D turbulence. However, the physical mechanism by which vortices and stellar systems achieve equilibrium is different. Whereas the organization of stars is relatively clear because of the attractive nature of gravity, the organization of point vortices at negative temperatures is much less intuitive. In the following section, we shall give a physical interpretation of this phenomenon in terms of a "systematic drift."

III. ELEMENTARY DERIVATION OF THE SYSTEMATIC DRIFT

A. Analogy with Brownian motion: The necessity of the drift

We shall first show the necessity of this drift by using an analogy with Brownian theory. The starting point of this analogy is to realize that the velocity of a point vortex can be decomposed in two terms, namely a smoothly varying function of position and time $\langle \mathbf{V} \rangle (\mathbf{r}, t)$ and a function $\boldsymbol{\mathcal{V}}(t)$ taking into account the "granularity" of the system and undergoing strong discontinuities. The total velocity of a point vortex can therefore be written

$$\mathbf{V} = \langle \mathbf{V} \rangle (\mathbf{r}, t) + \boldsymbol{\mathcal{V}}(t). \tag{30}$$

The velocity $\langle \mathbf{V} \rangle (\mathbf{r}, t)$ reflects the influence of the system as a *whole* and is generated by the mean vorticity $\langle \omega \rangle (\mathbf{r}, t)$ according to the Biot and Savart formula:

$$\langle \mathbf{V} \rangle (\mathbf{r}, t) = -\frac{1}{2\pi}\mathbf{z} \times \int \frac{\mathbf{r}' - \mathbf{r}}{|\mathbf{r}' - \mathbf{r}|^2} \langle \omega \rangle (\mathbf{r}', t) d^2\mathbf{r}'. \tag{31}$$

The fluctuation $\boldsymbol{\mathcal{V}}(t)$ arises from the difference between the exact distribution of the point vortices $\omega_{\text{exact}}(\mathbf{r}, t)$ and their "smoothed-out" distribution $\langle \omega \rangle (\mathbf{r}, t)$. It is on account of these fluctuations that the velocity of the test vortex will depart from its mean-field value $\langle \mathbf{V} \rangle$. The velocity fluctuation $\mathcal{V}$, of order γ/d (where $d \sim n^{-1/2}$ is the intervortex distance), is much smaller than the average velocity $\langle V \rangle$, of order $n\gamma R$ (where R is the domain size), but this term has a cumulative effect that gives rise to a process of diffusion. It makes sense, therefore, to introduce a stochastic description of the vortex motion such as that for colloidal suspensions in a liquid [17] or stars in globular clusters [8]. However, contrary to the ideal Brownian motion, point vortex systems have relatively long correlation times. This makes the study much more complicated than usual and a technical study of Sec. IV is required. In order to gain some physical insight into the problem, we shall ignore this difficulty for the moment and describe the system by traditional stochastic processes.

According to Eq. (30), we would naively expect that the evolution of the density probability $P(\mathbf{r}, t)$ would be governed by a diffusion equation of the form

$$\frac{\partial P}{\partial t} + \langle \mathbf{V} \rangle \boldsymbol{\nabla} P = D\Delta P. \tag{32}$$

This would in fact be the case for a passive particle having no retroaction on the vortices or when the distribution of vortices is uniform, as in [18,13,19]. However, this diffusion equation cannot be valid when the system is inhomogeneous. A first apparent reason is that Eq. (32) does not converge towards the Boltzmann distribution (26) when $t \to +\infty$. Another related difficulty is that Eq. (32) does not conserve energy. It seems, therefore, that a term is missing to act against the diffusion.

These problems are similar to those encountered in Brownian theory or for stellar systems. They have tradition-

ally been solved by introducing a *dynamical friction* in order to compensate for the effect of diffusion. The occurrence of this frictional force is a manifestation of the "fluctuation-dissipation" theorem in statistical mechanics. In the present context, the dynamical friction is replaced by a *systematic drift* of the vortices. We must therefore rewrite the decomposition (30) in the form

$$\mathbf{V} = \langle \mathbf{V} \rangle - \xi \boldsymbol{\nabla} \psi + \boldsymbol{\mathcal{V}}(t), \tag{33}$$

where ξ is the drift coefficient. In Sec. III B, we shall give a physical justification for the existence of the drift in terms of a polarization process, and in Sec. IV we shall derive this term directly from the Liouville equation by using projection-operator techniques. The importance of this drift was first pointed out by Chavanis [5] using a thermal bath approximation and a linear-response theory. The drift term must be calculated by resorting to relatively elaborate techniques, but it is remarkable that a general relationship between ξ and D can be obtained without analyzing at any point the details of the "subdynamics."

According to Eq. (33), the equation of motion for a point vortex can be written in the form

$$\frac{d\mathbf{r}}{dt} = \langle \mathbf{V} \rangle - \xi \boldsymbol{\nabla} \psi + \boldsymbol{\mathcal{V}}(t). \tag{34}$$

Since the velocity $\boldsymbol{\mathcal{V}}(t)$ undergoes strong discontinuities, the trajectory $\mathbf{r}(t)$ of the point vortex is not differentiable. Therefore, Eq. (34) must be viewed as a stochastic equation analogous to the Langevin equation in the ordinary Brownian theory. Let Δt be an interval of time long compared to the fluctuation time but short at the scale on which the physical parameters change appreciably. The variation in the position $\Delta \mathbf{r}$ of the particle during Δt is given by

$$\Delta \mathbf{r} = \langle \mathbf{V} \rangle \Delta t - \xi \boldsymbol{\nabla} \psi \Delta t + \mathbf{B}(\Delta t), \tag{35}$$

where

$$\mathbf{B}(\Delta t) = \int_{t}^{t+\Delta t} \boldsymbol{\mathcal{V}}(t') dt'. \tag{36}$$

Each fluctuation $\boldsymbol{\mathcal{V}}$ produces a small displacement $\delta \mathbf{r}$ but the repeated action of these fluctuations produces a net displacement of the same order as the drift $\xi \boldsymbol{\nabla} \psi$. To determine the probability $w[\mathbf{B}(\Delta t)]$ that the fluctuations produce a displacement $\mathbf{B}(\Delta t)$ during the time interval Δt, we first divide the interval $(t, t+\Delta t)$ into a succession of discrete increments in position and observe that $\mathbf{B}(\Delta t)$ is a sum of N random variables $T(\mathcal{V}_i)\mathcal{V}_i$, where $T(\mathcal{V})$ characterizes the typical duration of the velocity fluctuation $\boldsymbol{\mathcal{V}}$. This is a problem of random walks where $\mathbf{B}(\Delta t)$ represents the distance reached after N steps. For large N's, the central-limit theorem leads to a Gaussian transition probability:

$$w[\mathbf{B}(\Delta t)] = \frac{1}{4\pi D \Delta t} e^{-B(\Delta t)^2/4D\Delta t} \tag{37}$$

with a diffusion coefficient

$$D = \tfrac{1}{4} \langle T(\mathcal{V}) \mathcal{V}^2 \rangle. \tag{38}$$

We now assume that the motion of a point vortex can be idealized by a Markov process, i.e., the probability at time $t + \Delta t$ depends on the probability at time t but not at earlier times. As indicated previously, this approximation is not completely correct in the case of point vortices, which have long correlation times. However, relaxing this hypothesis would involve more intricate equations (see Sec. IV) and we shall ignore this difficulty for the moment. We write, therefore,

$$P(\mathbf{r}, t+\Delta t) = \int P(\mathbf{r} - \Delta \mathbf{r}, t) w(\mathbf{r} - \Delta \mathbf{r}|\Delta \mathbf{r}) d^2(\Delta \mathbf{r}), \tag{39}$$

where $w(\mathbf{r} - \Delta \mathbf{r}|\Delta \mathbf{r})$ is the probability for a point vortex located in $\mathbf{r} - \Delta \mathbf{r}$ to suffer an increment of position $\Delta \mathbf{r}$ during Δt. Expanding $P(\mathbf{r}, t+\Delta t)$, $P(\mathbf{r} - \Delta \mathbf{r}, t)$, and $w(\mathbf{r} - \Delta \mathbf{r}|\Delta \mathbf{r})$ in the form of Taylor series, we arrive at the Fokker-Planck equation in its general form:

$$\frac{\partial P}{\partial t} \Delta t = -\sum_{i} \frac{\partial}{\partial r_i} (P \langle \Delta r_i \rangle)$$
$$+ \frac{1}{2} \sum_{i,j} \frac{\partial^2}{\partial r_i \partial r_j} (P \langle \Delta r_i \Delta r_j \rangle), \tag{40}$$

where

$$\langle \Delta r_i \rangle = \int \Delta r_i w(\mathbf{r}|\Delta \mathbf{r}) d^2(\Delta \mathbf{r}), \tag{41}$$

$$\langle \Delta r_i \Delta r_j \rangle = \int \Delta r_i \Delta r_j w(\mathbf{r}|\Delta \mathbf{r}) d^2(\Delta \mathbf{r}). \tag{42}$$

According to Eq. (37), the transition probability from $\mathbf{r}$ to $\mathbf{r} + \Delta \mathbf{r}$ is given by

$$w(\mathbf{r}|\Delta \mathbf{r}) = \frac{1}{4\pi D \Delta t} \exp\left\{ -\frac{[\Delta \mathbf{r} - (\langle \mathbf{V} \rangle - \xi \boldsymbol{\nabla} \psi)\Delta t]^2}{4D\Delta t} \right\}. \tag{43}$$

With Eq. (43), the moments (41) and (42) can be easily evaluated, yielding

$$\langle \Delta \mathbf{r} \rangle = (\langle \mathbf{V} \rangle - \xi \boldsymbol{\nabla} \psi) \Delta t, \quad \langle \Delta r_i \Delta r_j \rangle = 2D\Delta t \delta_{ij}. \tag{44}$$

Substituting these results in the general Fokker-Planck equation (40), we find that

$$\frac{\partial P}{\partial t} + \langle \mathbf{V} \rangle \boldsymbol{\nabla} P = \boldsymbol{\nabla}(D \boldsymbol{\nabla} P + \xi P \boldsymbol{\nabla} \psi). \tag{45}$$

We had previously introduced this equation in Ref. [5] using phenomenological arguments. The physical interpretation of each term is straightforward. The left-hand side (which can be written dP/dt) is an advection term due to the smooth mean-field velocity $\langle \mathbf{V} \rangle$. The right-hand side can be written as the divergence of a current $-\boldsymbol{\nabla} \cdot \mathbf{J}$ and is the sum of two terms: the first term is a diffusion due to the erratic motion of

the vortices caused by the fluctuations $\mathcal{V}$; the second term accounts for the systematic drift of the vortices. At equilibrium, the drift precisely balances random scatterings and the distribution (26) is settled. More precisely, the condition that the Maxwell-Boltzmann statistics (26) satisfies Eq. (45) identically requires that D and ξ be related according to the relation

$$\xi = D\beta\gamma, \tag{46}$$

which is a generalization of the Einstein formula to the case of point vortices. A more rigorous justification of this relation will be given in Sec. IV, where the diffusion coefficient and the drift term are calculated explicitly.

B. Systematic drift: The result of a polarization process

According to the previous discussion, the relaxation of a point vortex towards statistical equilibrium can be described by a Fokker-Planck equation,

$$\frac{\partial P}{\partial t} + \langle \mathbf{V} \rangle \boldsymbol{\nabla} P = \boldsymbol{\nabla}[D(\boldsymbol{\nabla} P + \beta\gamma P \boldsymbol{\nabla}\psi)] \tag{47}$$

involving a diffusion term $-D\boldsymbol{\nabla}P$ and a drift term

$$\langle \mathbf{V} \rangle_{\text{drift}} = -D\beta\gamma\boldsymbol{\nabla}\psi. \tag{48}$$

The drift is normal to the mean-field velocity $\langle \mathbf{V} \rangle = -\mathbf{z} \times \boldsymbol{\nabla}\psi$ of the vortices, and its direction, depending on the sign of β, has important physical implications. To fix the ideas, let us assume that all point vortices have positive circulation (the opposite case leads to the same conclusion). Due to the mean-field velocity, a particular point vortex rotates anticlockwise. At negative temperatures, the drift is directed to its left and the vortex is *attracted* to the center of the domain. On the contrary, at positive temperatures, the drift is directed to its right and the vortex is *rejected* against the boundary. Therefore, the effect of the drift is consistent with the Onsager thermodynamical approach and it provides, in addition, a physical mechanism to understand the organization of point vortices at negative temperatures. For $\beta=0$, the medium is homogeneous and there is no drift. Equation (47) reduces to a pure diffusion equation like in Refs. [18,13,19]. Therefore, the drift occurs only in the presence of a background shear.

In stellar systems, the relaxation of the distribution function $f(\mathbf{r}, \mathbf{v}, t)$ is usually described by the Kramers-Chandrasekhar equation:

$$\frac{\partial f}{\partial t} + \mathbf{v}\frac{\partial f}{\partial \mathbf{r}} + \langle \mathbf{F} \rangle \frac{\partial f}{\partial \mathbf{v}} = \frac{\partial}{\partial \mathbf{v}}\left\{D\left(\frac{\partial f}{\partial \mathbf{v}} + \beta m f \mathbf{v}\right)\right\}, \tag{49}$$

which is a particular Fokker-Planck equation with a structure analogous to Eq. (47). In this analogy, the dynamical friction experienced by a star as a result of close encounters [8],

$$\langle \mathbf{F} \rangle_{\text{friction}} = -D\beta m \mathbf{v}, \tag{50}$$

is the counterpart of the systematic drift (48) experienced by a point vortex in two-dimensional turbulence. The dynamical

friction can be viewed as the drag exerted on a test star by the wake it induces in the field stars, as in a polarization process. We can use a similar approach to understand the origin of the drift. Let us consider a collection of N point vortices at statistical equilibrium with inverse temperature β. When $\beta < 0$, the density of these "field" vortices decreases from the center to the periphery of the domain. A "test" vortex moving through this medium locally modifies the vorticity distribution and produces a polarization cloud that amounts typically to a rotation of the surrounding vortices. This creates an excess of density behind it and a deficit of density in front of it. Therefore, the retroaction of the field vortices leads to a drift of the test vortex directed inward. We reach the opposite conclusion if the temperature is positive. When the system is homogeneous ($\beta = 0$ in a domain with no special symmetry), the polarization cloud induced by the test vortex has no effect and the drift cancels out. Therefore, a homogeneous system of point vortices remains homogeneous [18,13,19].

C. The maximum entropy production principle

In Sec. IV, we shall derive the Fokker-Planck equation (47) directly from the Liouville equation. However, we want to show first that the general structure of this equation can be understood from relatively simple thermodynamic arguments.

Let us rewrite Eq. (47) in the form

$$\frac{\partial \langle \omega \rangle}{\partial t} + \langle \mathbf{u} \rangle \boldsymbol{\nabla} \langle \omega \rangle = -\boldsymbol{\nabla} \cdot \mathbf{J}, \tag{51}$$

where $\mathbf{J}$ is an unknown diffusion current. This equation conserves the circulation (24) provided that $\mathbf{J} \cdot \hat{\mathbf{n}} = 0$ on the domain boundary ($\hat{\mathbf{n}}$ is a unit vector normal to the boundary). The problem at hand consists in determining the expression for $\mathbf{J}$. Its exact expression depends on microscopic processes and is therefore difficult to capture. However, it is easy to write down some macroscopic constraints that it must satisfy. These constraints are provided by the first and second principles of thermodynamics, namely the conservation of energy and the increase of entropy. We shall find that these constraints are very stringent and determine completely the structure of the diffusion current.

Taking the time derivative of Eqs. (20) and (22) and substituting for Eq. (51), we obtain the constraints

$$\dot{E} = \int \mathbf{J} \cdot \boldsymbol{\nabla}\psi d^2\mathbf{r} = 0 \tag{52}$$

and

$$\dot{S} = -\frac{1}{\gamma}\int \mathbf{J} \cdot \boldsymbol{\nabla} \ln\langle\omega\rangle d^2\mathbf{r} \geq 0. \tag{53}$$

We shall now introduce an optimization procedure known as the maximum entropy production principle (MEPP). This principle was introduced initially in the context of 2D turbulence by Robert and Sommeria [20], but its domain of applicability is very general and concerns, for example, the case

of stellar systems [9,21]. This principle states that "out of equilibrium, the system evolves so as to maximize its rate of entropy production $\dot{S}$ while accounting for all the constraints imposed by the dynamics, in particular the conservation of energy $\dot{E}=0$." There is no precise justification for this principle and it is important therefore to confront the MEPP with more rigorous methods, such as those of Sec. IV, to determine its domain of validity. In any case, the MEPP can be considered as a convenient tool to build relaxation equations that are mathematically well-behaved and that can serve as numerical algorithms to calculate maximum entropy states.

We seek, therefore, the optimal diffusion current **J**, which maximizes the rate of entropy production $\dot{S}$ at fixed energy. In order to avoid the unphysical solution $|\mathbf{J}|\rightarrow+\infty$ with $\mathbf{J}\cdot\nabla\psi=0$, we impose the additional constraint

$$J^2 \leq C(\mathbf{r},t), \tag{54}$$

where C is an upper bound which must exist but is not known. The solution of the optimization problem is

$$\mathbf{J}=-D[\nabla\langle\omega\rangle+\beta(t)\gamma\langle\omega\rangle\nabla\psi], \tag{55}$$

where $\beta(t)$ and $D(\mathbf{r},t)$ are Lagrange multipliers associated with the constraints (52) and (54). When substituted in Eq. (51), we obtain

$$\frac{\partial\langle\omega\rangle}{\partial t}+\langle\mathbf{V}\rangle\nabla\langle\omega\rangle=\nabla(D[\nabla\langle\omega\rangle+\beta(t)\gamma\langle\omega\rangle\nabla\psi]). \tag{56}$$

This equation has the same form as the Fokker-Planck equation (47). Here, the diffusion term arises from the variations of entropy $\delta\dot{S}$ and the drift term is necessary to conserve energy. Note that the Einstein formula (46) is automatically satisfied by this variational approach.

The time evolution of the inverse temperature $\beta(t)$ is determined by the conservation of energy. Substituting the diffusion current (55) in the constraint (52), we find

$$\beta(t)=-\frac{\displaystyle\int D\nabla\langle\omega\rangle\cdot\nabla\psi d^2\mathbf{r}}{\displaystyle\int D\gamma\langle\omega\rangle(\nabla\psi)^2 d^2\mathbf{r}}. \tag{57}$$

We can also check that the entropy monotonically increases during the relaxation provided that $D\geq 0$. Indeed, using Eqs. (53), (55), and (52), we can easily establish that

$$\dot{S}=\int\frac{J^2}{D\gamma\langle\omega\rangle}d^2\mathbf{r}\geq 0. \tag{58}$$

At equilibrium, $\mathbf{J}=0$ and we recover the Boltzmann distribution (26).

Note that the optimal current (55) can be written $\mathbf{J}=\chi\nabla\alpha$, where $\alpha=\ln\langle\omega\rangle+\beta\gamma\psi$ is a "generalized potential" that is uniform at equilibrium. Therefore, the MEPP can be viewed as a variational formulation of the linear thermodynamics of Onsager, which relates the diffusion currents to the

gradients of generalized potentials. However, the MEPP gives a more elegant approach to the problem and, conceding a real importance to the constraints, it is easier to implement in more complicated situations [20,9,22]. In addition, it shows that the structure of the relaxation is determined by purely thermodynamic arguments. All explicit reference to the subdynamics is encapsulated in the diffusion coefficient, which is left unspecified (it appears as a Lagrange multiplier related to an unknown bound on the diffusion current). It must therefore be calculated with a more microscopic model such as the one of Sec. IV.

IV. RELAXATION OF A POINT VORTEX IN A THERMAL BATH

A. The Liouville equation

Let us consider a collection of $N+1$ point vortices with identical circulation γ. We select one of these vortices, for example point vortex 0, and call it the "test vortex." The other vortices $1,\ldots,N$ will be refered to as the "field vortices." Let $\mu(\mathbf{r},\mathbf{r}_1,\ldots,\mathbf{r}_N,t)$ denote the $(N+1)$-particle distribution of the system, i.e.,

$$\mu(\mathbf{r},\mathbf{r}_1,\ldots,\mathbf{r}_N,t)d^2\mathbf{r}d^2\mathbf{r}_1\cdots d^2\mathbf{r}_N$$

represents the probability that point vortex 0 will be in the cell $(\mathbf{r},\mathbf{r}+d\mathbf{r})$, point vortex 1 in the cell $(\mathbf{r}_1,\mathbf{r}_1+d\mathbf{r}_1)\ldots$, and point vortex N in the cell $(\mathbf{r}_N,\mathbf{r}_N+d\mathbf{r}_N)$ at time t. The $(N+1)$-particle distribution function $\mu(t)$ satisfies the Liouville equation

$$\frac{\partial\mu}{\partial t}+\sum_{i=0}^{N}\mathbf{V}_i\frac{\partial\mu}{\partial\mathbf{r}_i}=0, \tag{59}$$

where $\mathbf{V}_i$ is the velocity of vortex i produced by the other vortices according to Eqs. (6) and (7). We also introduce the one- and N-particle distribution functions defined by

$$P(\mathbf{r},t)=\int\mu(\{\mathbf{r}_k\},t)\prod_{k=1}^{N}d^2\mathbf{r}_k, \tag{60}$$

$$\mu_{\text{sys}}(\mathbf{r}_1,\ldots,\mathbf{r}_N,t)=\int\mu(\{\mathbf{r}_k\},t)d^2\mathbf{r}. \tag{61}$$

The physical picture that we have in mind is that the test vortex evolves in a "bath" of field vortices. Therefore, we rewrite the distribution function μ in the suggestive form

$$\mu(\mathbf{r},\mathbf{r}_1,\ldots,\mathbf{r}_N,t)=P(\mathbf{r},t)\mu_{\text{sys}}(\mathbf{r}_1,\ldots,\mathbf{r}_N,t)$$
$$+\mu_I(\mathbf{r},\mathbf{r}_1,\ldots,\mathbf{r}_N,t), \tag{62}$$

where μ_I reflects the effect of correlations between the test vortex and the field vortices. Physically, this term accounts for the polarization process described qualitatively in Sec. III B.

The Liouville equation (59) provides the correct starting point for the analysis of the dynamics of our vortex system. However, when N is large, this equation contains much more information than one can interpret. Consequently, what one

would like to do is to describe the system in some average sense by a one-particle distribution function. In the previous sections, we have indeed derived heuristically some differential equation satisfied by this distribution function on the basis of stochastic arguments. We shall now discuss the connection of such heuristic theories with a more microscopic description of the system.

B. The projection operator formalism

Our first objective is to derive some *exact* kinetic equations satisfied by $P(\mathbf{r},t)$ and $\mu_{\text{sys}}(\mathbf{r}_1,\dots,\mathbf{r}_N,t)$. This can be achieved by using the projection operator formalism developed by Willis and Picard [6]. This formalism was also used by Kandrup [7] in the context of stellar dynamics to derive a generalized Landau equation describing the time evolution of the distribution function of stars in an inhomogeneous medium. We shall just recall the main steps of the theory. More details can be found in the original paper of Willis and Picard [6] and in Kandrup [7]. To have similar notations, we set $x \equiv \{\mathbf{r}\}$ and $y \equiv \{\mathbf{r}_1,\dots,\mathbf{r}_N\}$. Then, Eq. (62) can be set in the form

$$\mu(x,y,t) = \mu_R(x,y,t) + \mu_I(x,y,t) \tag{63}$$

with

$$\mu_R(x,y,t) = f(x,t)g(y,t), \tag{64}$$

where we have written $f(x,t) \equiv P(\mathbf{r},t)$ and $g(y,t) \equiv \mu_{\text{sys}}(\mathbf{r}_1,\dots,\mathbf{r}_N,t)$. The Liouville equation is also cast in the form

$$\frac{\partial \mu}{\partial t} = -iL\mu = -i(L_0 + L_{\text{sys}} + L')\mu, \tag{65}$$

where L_0 and L_{sys} act, respectively, only on the variables x and y, whereas the interaction Liouvillian L' acts upon both x and y (the complex number i is here purely formal and has been introduced only to have the same notations as Refs. [6,7]).

Following Willis and Picard, we introduce the time-dependent projection operator,

$$P(x,y,t) = g(y,t)\int dy + f(x,t)\int dx$$

$$- f(x,t)g(y,t)\int dx \int dy. \tag{66}$$

We can easily check that

$$P(x,y,t)\mu(x,y,t) = \mu_R(x,y,t), \tag{67}$$

$$[1 - P(x,y,t)]\mu(x,y,t) = \mu_I(x,y,t). \tag{68}$$

We also verify that P is a projection in the sense that $P^2(t) = P(t)$. Applying P and $1-P$ on the Liouville equation (65), we obtain the coupled equations

$$\partial_t\mu_R(x,y,t) = -iPL\mu_R - iPL\mu_I \tag{69}$$

and

$$\partial_t\mu_I(x,y,t) = -i(1-P)L\mu_R - i(1-P)L\mu_I. \tag{70}$$

These equations should be compared with Eqs. (8) and (9), which appear in the quasilinear theory of 2D turbulence [23]. In the present context, Eqs. (69) and (70) describe the separation between a "macrodynamics" and a "subdynamics." In the quasilinear theory, Eqs. (8) and (9) describe the evolution of the "coarse-grained" and "fine-grained" components of the vorticity.

Introducing the Greenian

$$\mathcal{G}(t,t') \equiv \exp\left\{-i\int_{t'}^{t} dt''[1 - P(t'')]L\right\}, \tag{71}$$

we can immediately write down a formal solution of Eq. (70), namely

$$\mu_I(x,y,t) = -\int_0^t dt'\,\mathcal{G}(t,t')i[1 - P(t')]L\mu_R(x,y,t'), \tag{72}$$

where we have assumed that initially the particles are uncorrelated so that $\mu_I(x,y,0) = 0$. Substituting for $\mu_I(x,y,t)$ from Eq. (72) in Eq. (69), we obtain

$$\partial_t\mu_R(x,y,t) = -iPL\mu_R - \int_0^t dt'\,P(t)L\mathcal{G}(t,t')$$

$$\times [1 - P(t')]L\mu_R(x,y,t'). \tag{73}$$

The integration over y will yield an equation describing the evolution of f. Using some mathematical properties of the projection operator (66), the final result can be set in the nice symmetrical form given by Willis and Picard [6]:

$$\partial_t f(x,t) + iL_0 f + i\langle L'\rangle_{\text{sys}} f$$

$$= -\int_0^t dt'\int dy\,\Delta_t L'\mathcal{G}(t,t')\Delta_{t'}L'g(y,t')f(x,t'), \tag{74}$$

where the notations stand for

$$\langle L'\rangle_{\text{sys}} = \int dy'\,L'(x,y')g(y',t), \tag{75}$$

$$\langle L'\rangle_0 = \int dx'\,L'(x',y)f(x',t), \tag{76}$$

$$\Delta_t L' = L' - \langle L'\rangle_{\text{sys}} - \langle L'\rangle_0. \tag{77}$$

Similarly, after integrating over x, we find the equation satisfied by g [6],

$$\partial_t g(y,t) + iL_{sys}g - i\langle L'\rangle_1 g$$

$$= -\int_0^t dt' \int dx \Delta_t L' \mathcal{G}(t,t') \Delta_{t'} L' g(y,t') f(x,t').$$

(78)

C. Application to the point vortex system

The previous theory is completely general and we now consider its application to a system of point vortices. Let us first rewrite the Liouville equation (59) in a form that separates the contribution of the test vortex from the contribution of the field vortices:

$$\frac{\partial \mu}{\partial t} + \sum_{i=1}^N \mathbf{V}(i\to 0)\frac{\partial \mu}{\partial \mathbf{r}} + \sum_{i=1}^N \mathbf{V}(0\to i)\frac{\partial \mu}{\partial \mathbf{r}_i}$$

$$+ \sum_{i=1}^N \sum_{j=1,j\neq i}^N \mathbf{V}(j\to i)\frac{\partial \mu}{\partial \mathbf{r}_i} = 0. \quad (79)$$

The different operators that arise in the decomposition (65) are

$$iL_0 = 0, \quad (80)$$

$$iL_{sys} = \sum_{i=1}^N \sum_{j\neq i,0} \mathbf{V}(j\to i)\frac{\partial}{\partial \mathbf{r}_i}, \quad (81)$$

$$iL' = \sum_{i=1}^N \left\{ \mathbf{V}(i\to 0)\frac{\partial}{\partial \mathbf{r}} + \mathbf{V}(0\to i)\frac{\partial}{\partial \mathbf{r}_i} \right\}. \quad (82)$$

The mean-field velocity created by the field vortex i on the test vortex is denoted by

$$\langle \mathbf{V}(i\to 0)\rangle = \int P(\mathbf{r}_i,t)\mathbf{V}(i\to 0)d^2\mathbf{r}_i. \quad (83)$$

Similarly,

$$\langle \mathbf{V}(0\to i)\rangle = \int P(\mathbf{r},t)\mathbf{V}(0\to i)d^2\mathbf{r} \quad (84)$$

denotes the mean-field velocity created by the test vortex on the field vortex i. Finally, the total mean-field velocity experienced by the test vortex is given by

$$\langle \mathbf{V}\rangle = \sum_{i=1}^N \langle \mathbf{V}(i\to 0)\rangle = \sum_{i=1}^N \langle \mathbf{V}(1\to 0)\rangle = N\langle \mathbf{V}(1\to 0)\rangle, \quad (85)$$

where the second equality follows from the identity of the point vortices.

We are now ready to evaluate the quantities (75), (76), and (77). After straightforward integration by parts, we find successively

$$i\langle L'\rangle_{sys} = \langle \mathbf{V}\rangle\frac{\partial}{\partial \mathbf{r}} = \sum_{i=1}^N \langle \mathbf{V}(i\to 0)\rangle\frac{\partial}{\partial \mathbf{r}}, \quad (86)$$

$$i\langle L'\rangle_0 = \sum_{i=1}^N \langle \mathbf{V}(0\to i)\rangle\frac{\partial}{\partial \mathbf{r}_i}, \quad (87)$$

$$i\Delta_t L' = \sum_{i=1}^N \{\mathbf{V}(i\to 0) - \langle \mathbf{V}(i\to 0)\rangle\}\frac{\partial}{\partial \mathbf{r}} + \sum_{i=1}^N \{\mathbf{V}(0\to i)$$

$$- \langle \mathbf{V}(0\to i)\rangle\}\frac{\partial}{\partial \mathbf{r}_i}. \quad (88)$$

Introducing the velocity fluctuations,

$$\boldsymbol{\mathcal{V}}(i\to 0) = \mathbf{V}(i\to 0) - \langle \mathbf{V}(i\to 0)\rangle, \quad (89)$$

$$\boldsymbol{\mathcal{V}}(0\to i) = \mathbf{V}(0\to i) - \langle \mathbf{V}(0\to i)\rangle, \quad (90)$$

we can rewrite our expression for $\Delta_t L'$ in the form

$$i\Delta_t L' = \sum_{i=1}^N \boldsymbol{\mathcal{V}}(i\to 0)\frac{\partial}{\partial \mathbf{r}} + \sum_{i=1}^N \boldsymbol{\mathcal{V}}(0\to i)\frac{\partial}{\partial \mathbf{r}_i}. \quad (91)$$

Substituting these results in Eq. (74), we obtain the following kinetic equation for the one-particle distribution function of a vortex system:

$$\frac{\partial P}{\partial t} + \langle \mathbf{V}\rangle\frac{\partial P}{\partial \mathbf{r}} = \int_0^t dt' \int \prod_{k=1}^N d^2\mathbf{r}_k \sum_{i=1}^N \boldsymbol{\mathcal{V}}(i\to 0)\frac{\partial}{\partial \mathbf{r}}$$

$$\times \mathcal{G}(t,t')\left\{ \sum_{j=1}^N \boldsymbol{\mathcal{V}}(j\to 0)\frac{\partial}{\partial \mathbf{r}} \right.$$

$$\left. + \sum_{j=1}^N \boldsymbol{\mathcal{V}}(0\to j)\frac{\partial}{\partial \mathbf{r}_j} \right\} P(\mathbf{r},t')\mu_{sys}(\{\mathbf{r}_k\},t')$$

(92)

or, alternatively,

$$\frac{\partial P}{\partial t} + \langle \mathbf{V}\rangle\frac{\partial P}{\partial \mathbf{r}}$$

$$= \frac{\partial}{\partial r^\mu}\int_0^t d\tau \int \prod_{k=1}^N d^2\mathbf{r}_k \sum_{i=1}^N \sum_{j=1}^N \mathcal{V}^\mu(i\to 0)\mathcal{G}(t,t-\tau)$$

$$\times \left(\mathcal{V}^\nu(j\to 0)\frac{\partial}{\partial r^\nu} + \mathcal{V}^\nu(0\to j)\frac{\partial}{\partial r_j^\nu} \right)$$

$$\times P(\mathbf{r},t-\tau)\mu_{sys}(\{\mathbf{r}_k\},t-\tau), \quad (93)$$

where the Greek indices refer to the components of $\boldsymbol{\mathcal{V}}$ in a fixed system of coordinates. We can note that Eq. (93) already shares some analogies with the Fokker-Planck equation of Sec. III. Indeed, the first term on the right-hand side corresponds to a diffusion and the second term to a drift. For a passive particle, $\mathcal{V}^\nu(0\to j) = 0$ and the drift cancels out, as expected.

D. The thermal bath approximation

Equation (93) is an *exact* differential equation for $P(\mathbf{r},t)$. However, this equation is not directly soluble since the unkown function $\mu_{\text{sys}}(\{\mathbf{r}_k\},t)$ is given by an equation of the form (78) depending in turn on $P(\mathbf{r},t)$. We have therefore to solve the coupled system (74)–(78). This system bears exactly the same information as the initial Liouville equation (65) and, without further simplification, is untractable.

To reduce the complexity of the problem, we shall implement a "thermal bath approximation." We assume that the field vortices are in statistical equilibrium with inverse temperature β_{eq}. Therefore, the N-particle distribution function $\mu_{\text{sys}}(\{\mathbf{r}_k\})$ can be approximated by a product of N one-particle distribution functions P_{eq} given by the Maxwell-Boltzmann statistics (26). In other words, we make the approximation

$$\mu_{\text{sys}}(\mathbf{r}_1,\ldots,\mathbf{r}_N,t)\simeq\mu_{\text{eq}}(\mathbf{r}_1,\ldots,\mathbf{r}_N) \tag{94}$$

with

$$\mu_{\text{eq}}(\mathbf{r}_1,\ldots,\mathbf{r}_N)=\prod_{k=1}^{N}P_{\text{eq}}(\mathbf{r}_k)=\prod_{k=1}^{N}A_k e^{-\beta_{\text{eq}}\gamma\psi_{\text{eq}}(\mathbf{r}_k)}, \tag{95}$$

where ψ_{eq} is solution of the Poisson equation (2) with the equilibrium vorticity $\langle\omega\rangle_{\text{eq}}=N\gamma P_{\text{eq}}$. Substituting explicitly for the Boltzmann distribution from Eq. (95) in Eq. (93), we obtain

$$\frac{\partial P}{\partial t}+\langle\mathbf{V}\rangle_{\text{eq}}\frac{\partial P}{\partial\mathbf{r}}$$

$$=\frac{\partial}{\partial r^{\mu}}\int_0^t d\tau\int\prod_{k=1}^{N}d^2\mathbf{r}_k\sum_{i=1}^{N}\sum_{j=1}^{N}\mathcal{V}^{\mu}(i\to0)\mathcal{G}(t,t-\tau)$$

$$\times\left(\mathcal{V}^{\nu}(j\to0)\frac{\partial}{\partial r^{\nu}}-\beta_{\text{eq}}\gamma\mathcal{V}^{\nu}(0\to j)\frac{\partial\psi_{\text{eq}}}{\partial r_j^{\nu}}\right)$$

$$\times P(\mathbf{r},t-\tau)\prod_{k=1}^{N}P_{\text{eq}}(\mathbf{r}_k), \tag{96}$$

where $\langle\ \rangle_{\text{eq}}$ denotes the average with respect to the equilibriun distribution P_{eq}. Explicating the action of the Greenian, we can rewrite our Eq. (96) in the form

$$\frac{\partial P}{\partial t}+\langle\mathbf{V}\rangle_{\text{eq}}\frac{\partial P}{\partial\mathbf{r}}=\frac{\partial}{\partial r^{\mu}}\int_0^t d\tau\int\prod_{k=1}^{N}d^2\mathbf{r}_k\sum_{i=1}^{N}\sum_{j=1}^{N}\mathcal{V}^{\mu}(i\to0,t)$$

$$\times\left(\mathcal{V}^{\nu}(j\to0,t-\tau)\frac{\partial}{\partial r^{\nu}}\right.$$

$$\left.-\beta_{\text{eq}}\gamma\mathcal{V}^{\nu}(0\to j,t-\tau)\frac{\partial\psi_{\text{eq}}}{\partial r_j^{\nu}}[\mathbf{r}_j(t-\tau)]\right)$$

$$\times P(\mathbf{r}(t-\tau),t-\tau)\prod_{k=1}^{N}P_{\text{eq}}[\mathbf{r}_k(t-\tau)], \tag{97}$$

where the retarded velocity $\mathcal{V}(j\to i,t-\tau)$ must be viewed as an explicit function of time. More precisely, $\mathcal{V}(j\to i,t-\tau)$ is a short-hand notation for $\mathcal{V}[\mathbf{r}_j(t-\tau)\to\mathbf{r}_i(t-\tau)]$, where $\mathbf{r}_i(t-\tau)$ denotes the position at time $t-\tau$ of the ith point vortex located in $\mathbf{r}_i\equiv\mathbf{r}_i(t)$ at time t. The trajectories of the point vortices between $t-\tau$ and t are determined by the complicated Greenian $\mathcal{G}(t,t-\tau)$ defined in Eq. (71). We need therefore to solve the exact Kirchhoff-Hamilton equations of motion (8). In fact, to a good approximation, we can consider that the point vortices are purely advected by the equilibrium mean-field velocity $\langle\mathbf{V}\rangle_{\text{eq}}$. Indeed, when $N\to\infty$, we have already indicated that the velocity fluctuation $\mathcal{V}$ is much smaller than the mean-field velocity $\langle\mathbf{V}\rangle_{\text{eq}}$. Therefore, we can replace the exact Greenian $\mathcal{G}$ by a smoother Greenian $\langle\mathcal{G}\rangle_{\text{eq}}$, which would be obtained if the point vortices were moving in the velocity field created by the equilibrium distribution function μ_{eq}. Formally, this Greenian is constructed with the averaged Liouville operator $\langle L\rangle_{\text{eq}}\equiv\Sigma_{i=0}^{N}\langle\mathbf{V}_i\rangle_{\text{eq}}(\partial/\partial\mathbf{r}_i)$.

In this approximation, the correlations involving two different vortex pairs vanish and the equation can be simplified considerably. Using the results of Appendix B [see, in particular, Eq. (B11)], we find

$$\frac{\partial P}{\partial t}+\langle\mathbf{V}\rangle_{\text{eq}}\frac{\partial P}{\partial\mathbf{r}}=\sum_{i=1}^{N}\frac{\partial}{\partial r^{\mu}}\int_0^t d\tau\int d^2\mathbf{r}_i V^{\mu}(i\to0,t)$$

$$\times\left(V^{\nu}(i\to0,t-\tau)\frac{\partial}{\partial r^{\nu}}\right.$$

$$-\beta_{\text{eq}}\gamma V^{\nu}(0\to i,t-\tau)$$

$$\left.\times\frac{\partial\psi_{\text{eq}}}{\partial r_i^{\nu}}[\mathbf{r}_i(t-\tau)]\right)$$

$$\times P(\mathbf{r}(t-\tau),t-\tau)P_{\text{eq}}(\mathbf{r}_i), \tag{98}$$

where we have used $P_{\text{eq}}[\mathbf{r}_i(t-\tau)]=P_{\text{eq}}(\mathbf{r}_i)$ since $P_{\text{eq}}=f(\psi_{\text{eq}})$ is constant along a streamline and the particles are assumed to follow the streamlines in a first approximation. Since the vortices are identical, we also have

$$\frac{\partial P}{\partial t}+\langle\mathbf{V}\rangle_{\text{eq}}\frac{\partial P}{\partial\mathbf{r}}=N\frac{\partial}{\partial r^{\mu}}\int_0^t d\tau\int d^2\mathbf{r}_1 V^{\mu}(1\to0,t)$$

$$\times\left(V^{\nu}(1\to0,t-\tau)\frac{\partial}{\partial r^{\nu}}\right.$$

$$-\beta_{\text{eq}}\gamma V^{\nu}(0\to1,t-\tau)\frac{\partial\psi_{\text{eq}}}{\partial r_1^{\nu}}$$

$$\left.\times[\mathbf{r}_1(t-\tau)]\right)$$

$$\times P(\mathbf{r}(t-\tau),t-\tau)P_{\mathrm{eq}}(\mathbf{r}_1). \tag{99}$$

Noting that the integral is dominated by the divergence of the product $V^\mu V^\nu$ when $\mathbf{r}_1 \to \mathbf{r}$, we can make a "local approximation" and replace $\partial \psi_{\mathrm{eq}}/\partial r_1^\nu(\mathbf{r}_1)$ and $P_{\mathrm{eq}}(\mathbf{r}_1)$ by their values taken in $\mathbf{r}$. For the same reason, we can neglect vortex images and replace the Kernel $\mathbf{V}(0 \to 1)$ by its singular part (7) satisfying $\mathbf{V}(0 \to 1) = -\mathbf{V}(0 \to 1)$. With these approximations, the kinetic equation (99) takes the form

$$\frac{\partial P}{\partial t} + \langle \mathbf{V} \rangle_{\mathrm{eq}} \frac{\partial P}{\partial \mathbf{r}} = N \frac{\partial}{\partial r^\mu} \int_0^t d\tau \int d^2\mathbf{r}_1 P_{\mathrm{eq}}(\mathbf{r}) V^\mu(1 \to 0,t)$$

$$\times V^\nu(1 \to 0,t-\tau) \left(\frac{\partial P}{\partial r^\nu}(\mathbf{r}(t-\tau),t-\tau) \right.$$

$$\left. + \beta_{\mathrm{eq}} \gamma P(\mathbf{r}(t-\tau),t-\tau) \frac{\partial \psi_{\mathrm{eq}}}{\partial r^\nu}[\mathbf{r}(t-\tau)] \right). \tag{100}$$

E. The Fokker-Planck equation

1. Unidirectionnal flow

We will now see how the preceding equation can be simplified for particular equilibrium flows. We shall first consider the case of a unidirectional flow $\langle \mathbf{V} \rangle_{\mathrm{eq}} = V_{\mathrm{eq}}(y)\hat{\mathbf{x}}$ produced by a vorticity distribution $\langle \omega \rangle_{\mathrm{eq}}(y)$. If we restrict ourselves to solutions of the form $P = P(y,t)$, the kinetic equation (100) becomes

$$\frac{\partial P}{\partial t} = N \frac{\partial}{\partial y} \int_0^t d\tau \int d^2\mathbf{r}_1 P_{\mathrm{eq}}(y) V^y(1 \to 0,t) V^y(1 \to 0,t-\tau)$$

$$\times \left(\frac{\partial P}{\partial y}(y,t-\tau) + \beta_{\mathrm{eq}} \gamma P(y,t-\tau) \frac{\partial \psi_{\mathrm{eq}}}{\partial y}(y) \right), \tag{101}$$

where we have used $y(t-\tau) = y(t) = y$ since the point vortices follow the streamlines of the equilibrium flow. This equation can be rewritten in the form

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial y} \int_0^t d\tau C(\tau) \left(\frac{\partial P}{\partial y}(y,t-\tau) + \beta_{\mathrm{eq}} \gamma P(y,t-\tau) \frac{\partial \psi_{\mathrm{eq}}}{\partial y} \right), \tag{102}$$

where

$$C(\tau) \equiv C^{yy}(\tau) = N \int d^2\mathbf{r}_1 V^y(1 \to 0,t) V^y(1 \to 0,t-\tau) P_{\mathrm{eq}}(y) \tag{103}$$

is the velocity autocorrelation function. In Appendix C 1, it is found that

$$C(\tau) = \frac{N\gamma^2}{8\pi} \ln N \frac{1}{1 + \frac{1}{4}\Sigma^2(y)\tau^2} P_{\mathrm{eq}}(y), \tag{104}$$

where

$$\Sigma(y) = -\frac{d}{dy} \langle V \rangle_{\mathrm{eq}}(y) \tag{105}$$

is the local shear of the flow, equal here to the vorticity. Note that the velocity autocorrelation function decays like τ^{-2} for $\tau \to +\infty$ [5].

Equation (102) is a non-Markovian equation since the probability $P(y,t)$ at time t depends on the probability $P(y,t-\tau)$ at earlier times through an integration over τ. Accordingly, the present study, which explicitly takes into account memory effects, is more general than the stochastic model presented in Sec. III A. However, if we implement a Markov approximation and replace $P(y,t-\tau)$ by $P(y,t)$, we recover the Fokker-Planck equation of Sec. III:

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial y} \left[D \left(\frac{\partial P}{\partial y} + \beta_{\mathrm{eq}} \gamma P \frac{\partial \psi_{\mathrm{eq}}}{\partial y} \right) \right] \tag{106}$$

with a diffusion coefficient

$$D \equiv D^{yy} = N \int_0^{+\infty} d\tau \int d^2\mathbf{r}_1 V^y(1 \to 0,t)$$

$$\times V^y(1 \to 0,t-\tau) P_{\mathrm{eq}}(y) \tag{107}$$

and a drift term

$$\eta_y \equiv -\langle V_y \rangle_{\mathrm{drift}} = \beta_{\mathrm{eq}} \gamma D \frac{\partial \psi_{\mathrm{eq}}}{\partial y}. \tag{108}$$

The drift coefficient is given by an Einstein relation as expected from the general considerations of Sec. III A. The diffusion coefficient is expressed as a Kubo formula, i.e., as the integral of the velocity correlation function (see Appendix B). In Appendix C 1, it is found that (see also Ref. [5])

$$D = \frac{1}{8}N\gamma^2 \frac{1}{|\Sigma(y)|} \ln N P_{\mathrm{eq}}(y). \tag{109}$$

The reason for the logarithmic divergence is explained in Appendix A and in Ref. [13].

2. Axisymmetrical flow

We now consider the case of an axisymmetrical equilibrium flow such that $\langle \mathbf{V} \rangle_{\mathrm{eq}} = \langle V \rangle_{\mathrm{eq}}(r)\hat{\mathbf{e}}_\theta$. This flow is generated by an equilibrium vorticity field $\langle \omega \rangle_{\mathrm{eq}}(r)$. If we restrict ourselves to solutions of the form $P = P(r,t)$, the kinetic equation (100) simplifies in

$$\frac{\partial P}{\partial t} = N \frac{1}{r} \frac{\partial}{\partial r} r \int_0^t d\tau \int d^2\mathbf{r}_1 P_{\mathrm{eq}}(r) V^{r(t)}(1 \to 0,t)$$

$$\times V^{r(t-\tau)}(1 \to 0,t-\tau)$$

$$\times \left(\frac{\partial P}{\partial r}(r,t-\tau) + \beta_{\mathrm{eq}} \gamma P(r,t-\tau) \frac{\partial \psi_{\mathrm{eq}}}{\partial r}(r) \right), \tag{110}$$

where we have used $r(t-\tau)=r(t)=r$ since the point vortices follow the streamlines of the equilibrium flow. Furthermore, $V^{r(t)}$ and $V^{r(t-\tau)}$ denote the radial components of $\mathbf{V}(1\to0)$ at times t and $t-\tau$ in a polar system of coordinates $(\hat{\mathbf{e}}_r(t),\hat{\mathbf{e}}_\theta(t))$ moving with the test vortex. Equation (110) can be rewritten

$$\frac{\partial P}{\partial t}=\frac{1}{r}\frac{\partial}{\partial r}r\int_0^t d\tau C(\tau)\left(\frac{\partial P}{\partial r}(r,t-\tau)+\beta_{\rm eq}\gamma P(r,t-\tau)\frac{\partial\psi_{\rm eq}}{\partial r}\right),$$

(111)

where $C(\tau)$ is the velocity autocorrelation function

$$C(\tau)=N\int d^2\mathbf{r}_1 V^{r(t)}(1\to0,t)V^{r(t-\tau)}(1\to0,t-\tau)P_{\rm eq}(r).$$

(112)

In Appendix C 2, it is found that

$$C(\tau)=\frac{N\gamma^2}{8\pi}\ln N\frac{1}{1+\frac{1}{4}\Sigma^2(r)\tau^2}P_{\rm eq}(r),$$

(113)

where

$$\Sigma(r)=r\frac{d}{dr}\left(\frac{\langle V\rangle_{\rm eq}(r)}{r}\right)$$

(114)

is the local shear of the flow.

If we ignore memory effects, we obtain the Fokker-Planck equation,

$$\frac{\partial P}{\partial t}=\frac{1}{r}\frac{\partial}{\partial r}\left[rD\left(\frac{\partial P}{\partial r}+\beta_{\rm eq}\gamma P\frac{\partial\psi_{\rm eq}}{\partial r}\right)\right]$$

(115)

with a diffusion coefficient

$$D=N\int_0^{+\infty}d\tau\int d^2\mathbf{r}_1 V^{r(t)}(1\to0,t)$$

$$\times V^{r(t-\tau)}(1\to0,t-\tau)P_{\rm eq}(r)$$

(116)

and a drift term

$$\eta_r=-\langle V_r\rangle_{\rm drift}=\beta_{\rm eq}\gamma D\frac{\partial\psi_{\rm eq}}{\partial r}.$$

(117)

Explicitly, the diffusion coefficient has the form (see Appendix C 2 and Ref. [5])

$$D=\tfrac{1}{8}N\gamma^2\frac{1}{|\Sigma(r)|}\ln NP_{\rm eq}(r).$$

(118)

3. The general case

We can show in the general case that the relaxation of the test vortex is described by the Fokker-Planck equation

$$\frac{\partial P}{\partial t}+\langle\mathbf{V}_{\rm eq}\rangle\boldsymbol{\nabla}P=\boldsymbol{\nabla}[D(\boldsymbol{\nabla}P+\beta_{\rm eq}\gamma P\boldsymbol{\nabla}\psi_{\rm eq})]$$

(119)

with a diffusion coefficient

$$D=\frac{\gamma}{8}\frac{1}{|\Sigma(\mathbf{r})|}\ln N\langle\omega\rangle_{\rm eq},$$

(120)

where $|\Sigma(\mathbf{r})|=2\sqrt{-\det(\Sigma)}$ is the local shear of the flow and and $\det(\Sigma)$ is the determinant of the stress tensor $\Sigma^{\mu\nu}$ (see Sec. V C). In the regions where the shear cancels out, our approximations clearly break down. In particular, we cannot calculate the Kubo integral by assuming that the vortices follow the streamlines of the equilibrium flow. This is because, for a local solid rotation, the vortices always remain at the same relative distance and the correlation time is infinite. In that case, it is necessary to take into account the dispersion of the vortices. An alternative derivation of the diffusion coefficient can be obtained by analyzing the statistics of velocity fluctuations created by a random distribution of point vortices [13]. When the differential rotation of the vortices is neglected [which corresponds to the opposite limit of that leading to Eq. (120)], we obtain (see Ref. [13] and Appendix A)

$$D\sim\gamma\sqrt{\ln N}.$$

(121)

Clearly, a more complete study should take into account simultaneously the effect of the shear and the dispersion of the vortices to match the two formulas (120) and (121).

For $t\to+\infty$, the distribution function $P(\mathbf{r},t)$ of the test vortex converges towards the Maxwell-Boltzmann statistics (26). The time of relaxation corresponds typically to the time needed by the test vortex to diffuse over a distance R, the system size. Therefore, $t_{\rm relax}\sim R^2/D$, where $D\sim\gamma\ln N$ is the order of magnitude of the diffusion coefficient given by Eq. (120). Using $\Gamma=N\gamma$ and introducing the dynamical time $t_D\sim\langle\omega\rangle^{-1}\sim R^2/\Gamma$, we obtain the estimate

$$t_{\rm relax}\sim\frac{N}{\ln N}t_D.$$

(122)

Since the statistical description is expected to yield relevant results for large N, we conclude that the relaxation of point vortices towards the Boltzmann distribution is a very slow process. It is plausible that a more violent relaxation may be at work in the system. This problem is discussed more specifically in the conclusion.

In the previous calculations, we have assumed that the distribution of the field vortices is given by the Maxwell-Boltzmann statistics (26), which corresponds to statistical equilibrium. In the case of an arbitrary background distribution $P_{\rm eq}$ the expression of the drift is

$$\langle\mathbf{V}\rangle_{\rm drift}=D\boldsymbol{\nabla}\ln P_{\rm eq},$$

(123)

where D is still given by Eq. (120). Since $D>0$, the drift is always directed along the density gradient. The estimate of the time of relaxation is not changed in this more general situation.

V. A GENERALIZED KINETIC EQUATION

In the preceding section, we have described the relaxation of a test vortex in a "bath" of field vortices at statistical

equilibrium. We would like now to relax this thermal bath approximation in order to obtain a generalized kinetic equation describing the evolution of the whole system.

A. The factorization hypothesis

If the vortices are initially decorrelated, then, for sufficiently short times, they will remain decorrelated. This means that the $(N+1)$-particle distribution function can be factorized in a product of $(N+1)$ one-particle distribution functions:

$$\mu(\mathbf{r},\mathbf{r}_1,\ldots,\mathbf{r}_N,t)=\prod_{k=0}^{N} P(\mathbf{r}_k,t). \qquad (124)$$

If we integrate the Liouville equation (59) on the positions of the N vortices $1,\ldots,N$ and use the factorization (124), we directly obtain

$$\frac{\partial P}{\partial t}+\langle\mathbf{V}\rangle\boldsymbol{\nabla} P=0. \qquad (125)$$

Therefore, for sufficiently short times, the average vorticity $\langle\omega\rangle$ satisfies the 2D Euler equation. However, at later times, the distribution function μ differs from the pure product (124) and the Euler equation does not provide a good approximation anymore. In Sec. IV A, we have determined an exact equation (93) satisfied by the one-particle distribution function at any time. This equation is not closed, however, since it involves the N-vortex distribution function μ_{sys}. We shall close the system by assuming that μ_{sys} can still be approximated by a product of N one-particle distribution functions:

$$\mu_{\mathrm{sys}}(\mathbf{r}_1,\ldots,\mathbf{r}_N,t)\simeq\prod_{k=1}^{N} P(\mathbf{r}_k,t), \qquad (126)$$

but, contrary to Sec. IV D, the one-particle distribution function $P(\mathbf{r}_k,t)$ is *not* ascribed to the equilibrium value P_{eq}. Physically, the decomposition (62) and (126) assumes that the correlations that develop between the vortices (term μ_I) are due uniquely to the polarization cloud imposed by each individual vortex. Without this polarization, the vortices would be uncorrelated (term μ_{sys}). In particular, this decomposition does not take into account three-body encounters, which can play a crucial role in the dynamics of vortices (in particular when the system is neutral and homogeneous, see Refs. [24,19]). These high-order correlations develop on longer time scales and may be neglected in a first approach.

The approximation (126) introduced in Eq. (93) leads to a generalized kinetic equation,

$$\frac{\partial P}{\partial t}+\langle\mathbf{V}\rangle\frac{\partial P}{\partial\mathbf{r}}$$

$$=\frac{\partial}{\partial r^{\mu}}\int\prod_{k=1}^{N} d^2\mathbf{r}_k\int_0^t d\tau\sum_{i=1}^{N}\sum_{j=1}^{N}\mathcal{V}^{\mu}(i\to0)\mathcal{G}(t,t-\tau)$$

$$\times\left(\mathcal{V}^{\nu}(j\to0)\frac{\partial}{\partial r^{\nu}}+\mathcal{V}^{\nu}(0\to j)\frac{\partial}{\partial r_j^{\nu}}\right)$$

$$\times P(\mathbf{r},t-\tau)\prod_{k=1}^{N} P(\mathbf{r}_k,t-\tau). \qquad (127)$$

Repeating the steps leading from Eq. (96) to Eq. (99), we obtain

$$\frac{\partial P}{\partial t}+\langle\mathbf{V}\rangle\frac{\partial P}{\partial\mathbf{r}}=N\frac{\partial}{\partial r^{\mu}}\int_0^t d\tau\int d^2\mathbf{r}_1 V^{\mu}(1\to0)_t$$

$$\times\left\{V^{\nu}(1\to0)P_1\frac{\partial P}{\partial r^{\nu}}\right.$$

$$\left.+V^{\nu}(0\to1)P\frac{\partial P_1}{\partial r_1^{\nu}}\right\}_{t-\tau}, \qquad (128)$$

where $P=P(\mathbf{r},t)$ and $P_1=P(\mathbf{r}_1,t)$. We also recall that between t and $t-\tau$, the trajectories of the particles are determined from the smooth velocity field created by the vorticity distribution $\langle\omega\rangle=N\gamma P(\mathbf{r},t)$. This non-Markovian integrodifferential equation is similar to the equation obtained by Chavanis [23] in two-dimensional turbulence using a quasilinear theory of the Euler-Poisson system. It is shown in Appendix D that this equation rigorously conserves angular momentum in a circular domain and linear impulse in a channel (or in an infinite domain). However, under this form, it is not possible to prove the conservation of energy and the H theorem. In our previous investigation of the problem [23], we have considered a somewhat crude approximation, which amounts to neglecting memory effects in Eq. (128). We beleived that this approximation would not introduce any significant error, but, as we shall see, we were wrong: such an approximation breaks the conservation of energy.

If we assume that the decorrelation time τ is short (which does not need to be the case) and we implement a strong Markov approximation, we obtain

$$\frac{\partial P}{\partial t}+\langle\mathbf{V}\rangle\frac{\partial P}{\partial\mathbf{r}}=\frac{N\tau}{2}\frac{\partial}{\partial r^{\mu}}\int d^2\mathbf{r}_1 V^{\mu}(1\to0)\left\{V^{\nu}(1\to0)P_1\frac{\partial P}{\partial r^{\nu}}\right.$$

$$\left.+V^{\nu}(0\to1)P\frac{\partial P_1}{\partial r_1^{\nu}}\right\}. \qquad (129)$$

In the case of an infinite domain, $\mathbf{V}(0\to1)=-\mathbf{V}(0\to1)$ and we have the further simplification

$$\frac{\partial P}{\partial t} + \langle \mathbf{V} \rangle \frac{\partial P}{\partial \mathbf{r}} = \frac{N\gamma^2}{8\pi^2} \tau \frac{\partial}{\partial r^\mu} \int d^2\mathbf{r}_1 K'^{\mu\nu}(\boldsymbol{\xi}) \left(P_1 \frac{\partial P}{\partial r^\nu} \right.$$

$$\left. - P \frac{\partial P_1}{\partial r_1^\nu} \right), \tag{130}$$

where

$$K'^{\mu\nu}(\boldsymbol{\xi}) = \frac{\xi_\perp^\mu \xi_\perp^\nu}{\xi^4} = \frac{\xi^2 \delta^{\mu\nu} - \xi^\mu \xi^\nu}{\xi^4} \tag{131}$$

and $\boldsymbol{\xi} = \mathbf{r}_1 - \mathbf{r}$ (we also recall that $\boldsymbol{\xi}_\perp$ is the vector $\boldsymbol{\xi}$ rotated by $+\pi/2$). To arrive at Eq. (131), we have explicitly used the form of the kernel (7), and to get the second equality we have used the fact that we are in two dimensions. Note that the symmetrical form of Eq. (130) is reminiscent of the Landau equation introduced in plasma physics and in stellar dynamics [7]. In this analogy, the position $\mathbf{r}$ of the vortices plays the role of the velocity $\mathbf{v}$ of the electric charges or stars. Therefore, we can directly infer the conservation of linear impulse $\mathbf{P}_\perp = \int \omega \mathbf{r} d^2\mathbf{r}$ and angular momentum $L = \int \omega r^2 d^2\mathbf{r}$, which play, respectively, the role of impulse $\mathbf{P} = \int f\mathbf{v}d^3\mathbf{v}$ and kinetic energy $K = \int f(v^2/2)d^3\mathbf{v}$ in plasma physics. In addition, we can prove a H theorem for the Boltzmann entropy (22) exactly like for the Landau equation. Finally, we can show that the solutions of Eq. (130) converge towards the Gaussian vortex (the equivalent of the Maxwellian distribution in plasma physics with $\mathbf{r}$ in place of $\mathbf{v}$):

$$P(\mathbf{r}) = A e^{-(1/2)\alpha\gamma(\mathbf{r} - \mathbf{r}_0)^2}, \tag{132}$$

which is the maximum entropy state at fixed circulation, angular momentum, and impulse. It is in general different from the Boltzmann distribution (26) with the relative stream function $\psi' = \psi + (\Omega/2)r^2 - \mathbf{U}_\perp \cdot \mathbf{r}$. This clearly indicates that Eq. (129) does *not* conserve the meanfield energy (20). It may happen, however, that the energy is approximately conserved if we start from an initial condition with a value of energy E_0 corresponding to $\beta \to 0$, $\Omega \to +\infty$, and $\alpha = \beta\Omega/2$ finite at equilibrium. In that case, Eq. (132) is the maximum entropy state at fixed Γ, L, and E_0 (the conservation of impulse can be satisfied trivially by taking the center of vorticity as the origin of our system of coordinates). However, if E differs from E_0 by a large amount, the kinetic equation (129) will not correctly describe the evolution of the system for late times.

Now, if we account properly for memory effects in Eq. (128), we can obtain a more general equation that guarantees in addition the conservation of energy and is therefore more satisfactory. In the case of an axisymmetrical flow, it is possible to calculate the memory function appearing in Eq. (128) explicitly if we assume that the correlation time is smaller (but not necessarily *much* smaller) than the typical time on which the average vorticity changes appreciably. In this approximation, the point vortices follow, between t and $t - \tau$, circular trajectories with angular velocity $\Omega(r,t) = \langle V_\theta \rangle (r,t)/r$, and Eq. (128) simplifies in (see Appendix E)

$$\frac{\partial P}{\partial t} = -\frac{N\gamma^2}{4r} \frac{\partial}{\partial r} \int_0^{+\infty} r_1 dr_1 \delta(\Omega - \Omega_1) \ln\left[1 - \left(\frac{r_<}{r_>} \right)^2 \right]$$

$$\times \left\{ \frac{1}{r} P_1 \frac{\partial P}{\partial r} - \frac{1}{r_1} P \frac{\partial P_1}{\partial r_1} \right\}, \tag{133}$$

where $\Omega = \Omega(r,t)$, $\Omega_1 = \Omega(r_1,t)$, and $r_>$ $(r_<)$ is the biggest (smallest) of r and r_1. The angular velocity is related to the vorticity by

$$\langle \omega \rangle = \frac{1}{r} \frac{\partial}{\partial r} (\Omega r^2). \tag{134}$$

Similarly, in the case of a unidirectional flow, we can assume that, between t and $t - \tau$, the point vortices follow linear trajectories with velocity $\langle \mathbf{V} \rangle = \langle V \rangle (y,t)\mathbf{e}_x$. This leads to the kinetic equation (see Appendix E)

$$\frac{\partial P}{\partial t} = \frac{N\gamma^2}{4} \frac{\partial}{\partial y} \int_{-\infty}^{+\infty} dy_1 \delta(V_1 - V) E_1\left(\frac{2|y_1 - y|}{L} \right)$$

$$\times \left(P_1 \frac{\partial P}{\partial y} - P \frac{\partial P_1}{\partial y_1} \right), \tag{135}$$

where $V = \langle V \rangle (y,t)$ and $V_1 = \langle V \rangle (y_1,t)$. The function $E_1(x)$ is the exponential integral and L is an upper cutoff necessary in that case (see Appendix E). We also recall that the average velocity is related to the vorticity by

$$\langle \omega \rangle = -\frac{\partial}{\partial y} \langle V \rangle. \tag{136}$$

We can remarkably propose an approximation of the general kinetic equation (128) which encompasses both the axisymmetric form (133) and the unidirectional form (135). Memory effects are not neglected, unlike in Eq. (130), but they are simplified in a way which preserves all the conservation laws of the system (as discussed below). We propose the generalized kinetic equation:

$$\frac{\partial P}{\partial t} + \langle \mathbf{V} \rangle \boldsymbol{\nabla} P = \frac{N\gamma^2}{8} \frac{\partial}{\partial r^\mu} \int d^2\mathbf{r}_1 K^{\mu\nu} \delta(\boldsymbol{\xi} \cdot \mathbf{v})$$

$$\times \left(P_1 \frac{\partial P}{\partial r^\nu} - P \frac{\partial P_1}{\partial r_1^\nu} \right) \tag{137}$$

with

$$K^{\mu\nu}(\boldsymbol{\xi}) = \frac{\xi_\perp^\mu \xi_\perp^\nu}{\xi^2} = \frac{\xi^2 \delta^{\mu\nu} - \xi^\mu \xi^\nu}{\xi^2} \tag{138}$$

and $\boldsymbol{\xi} = \mathbf{r}_1 - \mathbf{r}$, $\mathbf{v} = \langle \mathbf{V} \rangle (\mathbf{r}_1, t) - \langle \mathbf{V} \rangle (\mathbf{r}, t)$. For specific applications, it may be necessary to introduce a shielding of the form (E39) in the interaction between vortices. This shielding arises naturally in geophysics in the "quasigeostrophic approximation." In that case, the tensor $K^{\mu\nu}$ is replaced by $(1/L)K_1(\xi/L)\xi_\perp^\mu \xi_\perp^\nu / \xi$, where K_1 is the modified Bessel function of first order and L is an upper cutoff (called the

Rossby radius in geophysics), which plays the same role as the Debye length in plasma physics.

Equation (137) is not exact, in a strict sense, yet it satisfies all the conservations laws of the vortex system as well as a H theorem (Sec. V B). This is very gratifying and can have important practical applications. It is remarkable that we can write down an approximate kinetic equation in the general case without being required to specify the trajectory of the point vortices between t and $t-\tau$. In fact, to arrive at Eq. (137), we have made implicitly two approximations. (i) We have assumed that the vorticity field does not change dramatically when we follow the vortices in their motion between t and $t-\tau$. (ii) After the time integration has been effected, the nonuniversal function $V^\nu(1\rightarrow0,t-\tau)$ gives rise to a logarithmic term, which has been replaced by 1 in the subsequent calculations. This term produces a sublogarithmic correction that is flow-dependent and that has been neglected. It is on account of this weak dependence that a general kinetic equation can be obtained. For axisymmetrical or unidirectionnal flows, Eq. (137) reduces to Eqs. (133) and (135) with $\ln(|r_1-r|/(r_1+r))$ and $K_0(|y-y_1|/L)$ instead of $\ln(1-(r_</r_>)^2)$ and $E_1(2|y-y_1|/L)$. This discrepency is not too severe (these functions have a similar logarithmic behavior) so our approximations are reasonable.

From Eqs. (133), (135), and (137), it is clear that the relaxation towards equilibrium is due to a phenomenon of resonance. Only the points $\mathbf{r}_1$ satisfying the condition $\boldsymbol{\xi}\cdot\mathbf{v}=\mathbf{0}$ with $\mathbf{r}_1\neq\mathbf{r}$ will contribute to the diffusion current in $\mathbf{r}$. In the axisymmetrical case, this condition of resonance reduces to $\Omega(r_1)=\Omega(r)$ and in the unidirectional case to $V(y_1)=V(y)$. These conditions of resonance had never been noticed previously. Further work on the subject will have to make these criteria more precise by computing explicitly "resonance lines" in two-dimensional real flows.

B. Conservation laws and H theorem

We now derive the conservation laws and the H theorem satisfied by Eq. (137). First of all, the conservation of the circulation is straightforward since Eq. (137) can be written in the form of a continuity equation (D1). To prove the conservation of angular momentum, we start from Eq. (D4), substitute for Eq. (137), permut the dummy variables $\mathbf{r}$ and $\mathbf{r}_1$, and add the resulting expressions. This yields

$$\dot{L}=\frac{N^2\gamma^3}{8}\int d^2\mathbf{r}d^2\mathbf{r}_1K^{\mu\nu}\xi^\mu\delta(\boldsymbol{\xi}\cdot\mathbf{v})\left(P_1\frac{\partial P}{\partial r^\nu}-P\frac{\partial P_1}{\partial r_1^\nu}\right).$$
(139)

But, from Eq. (138), we immediately verify that

$$K^{\mu\nu}\xi^\mu=0.$$
(140)

This proves the conservation of angular momentum. We can prove the conservation of linear impulse in a similar manner. Starting from Eq. (D16), substituting for Eq. (137), permuting the dummy variables $\mathbf{r}$ and $\mathbf{r}_1$, and adding the two result-

ing expressions yields $\dot{\mathbf{P}}=\mathbf{0}$. For the conservation of energy, we start from Eq. (52) and follow the by now familiar procedure. This yields

$$\dot{E}=\frac{N^2\gamma^3}{16}\int d^2\mathbf{r}d^2\mathbf{r}_1K^{\mu\nu}v_\perp^\mu\delta(\boldsymbol{\xi}\cdot\mathbf{v})\left(P_1\frac{\partial P}{\partial r^\nu}-P\frac{\partial P_1}{\partial r_1^\nu}\right).$$
(141)

Considering the form of the tensor (138), we have

$$K^{\mu\nu}v_\perp^\mu=\frac{\xi_\perp^\nu}{\xi^2}(\boldsymbol{\xi}\cdot\mathbf{v}).$$
(142)

When substituted in Eq. (141), we see that the occurrence of the δ function in the kinetic equation implies $\dot{E}=0$. Finally, for the rate of entropy production, we have, according to Eqs. (53) and (137),

$$\dot{S}=\frac{N^2\gamma^2}{8}\int d^2\mathbf{r}d^2\mathbf{r}_1\frac{1}{PP_1}P_1\frac{\partial P}{\partial r^\mu}K^{\mu\nu}\delta(\boldsymbol{\xi}\cdot\mathbf{v})$$

$$\times\left(P_1\frac{\partial P}{\partial r^\nu}-P\frac{\partial P_1}{\partial r_1^\nu}\right).$$
(143)

Permuting the dummy variables $\mathbf{r}$ and $\mathbf{r}_1$ and adding the resulting expression to Eq. (143), we get

$$\dot{S}=\frac{N^2\gamma^2}{16}\int d^2\mathbf{r}d^2\mathbf{r}_1\frac{1}{PP_1}\delta(\boldsymbol{\xi}\cdot\mathbf{v})$$

$$\times\left(P_1\frac{\partial P}{\partial r^\mu}-P\frac{\partial P_1}{\partial r_1^\mu}\right)K^{\mu\nu}\left(P_1\frac{\partial P}{\partial r^\nu}-P\frac{\partial P_1}{\partial r_1^\nu}\right).$$
(144)

Now, for any vector, $A^\mu K^{\mu\nu}A^\nu=(\mathbf{A}\cdot\boldsymbol{\xi}_\perp)^2/\xi^2\geqslant0$. This proves a H theorem ($\dot{S}\geqslant0$) for our kinetic equation (137). It should be emphasized that the conservation laws and the H theorem result from the *symmetry* of the kinetic equation (and the condition of resonance) and not from formal Lagrange multipliers as in the thermodynamical approach of Sec. III C. In addition, the H theorem is *proved* by our approach instead of beeing postulated. This is more satisfying on physical grounds.

It remains for one to show that the Boltzmann distribution

$$P=Ae^{-\beta\gamma[\psi+(\Omega/2)r^2-\mathbf{U}_\perp\cdot\mathbf{r}]}$$
(145)

is a stationary solution of Eq. (137). Noting that

$$\frac{\partial P}{\partial r^\nu}=-\beta\gamma\left(\frac{\partial\psi}{\partial r^\nu}+\Omega r^\nu-U_\perp^\nu\right)P,$$
(146)

we have successively

$$K^{\mu\nu}\left(P_1\frac{\partial P}{\partial r^\nu}-P\frac{\partial P_1}{\partial r_1^\nu}\right)=\beta\gamma PP_1K^{\mu\nu}(v_\perp^\nu+\Omega\xi^\nu)$$

$$=\beta\gamma PP_1\frac{\xi_\perp^\mu}{\xi^2}(\boldsymbol{\xi}\cdot\mathbf{v}),\qquad(147)$$

where we have used Eqs. (140) and (142). When substituted in Eq. (137), we find that the right-hand side cancels out due to the δ function. The advective term is also zero since $P=f(\psi')$. Therefore, the distribution (145) is a stationary solution of Eq. (137). Note, however, that this is not the only solution. Any stationary solution satisfying $\boldsymbol{\xi}\cdot\mathbf{v}\neq\mathbf{0}$ for any couple of points $\mathbf{r},\mathbf{r}_1$ (with $\mathbf{r}\neq\mathbf{r}_1$) is a solution of Eq. (137). Physically, this implies that the system needs sufficiently strong resonances to relax towards the maximum entropy state. If this is not realized, it will be frozen in a sort of "metastable" equilibrium. This may explain why the maximum entropy state is not always reached in two-dimensional turbulence. For example, for a unidirectional flow with $\langle\omega\rangle$ positive or negative everywhere, the velocity field is monotonous [see Eq. (136)] and the condition of resonance cannot be satisfied. The evolution of the system will require nontrivial correlations between point vortices that are not taken into account in the present approch. One would need to replace the factorization hypothesis (126) by a product of two-point or three-point correlation functions. However, it is plausible that these correlations develop on a very long time scale, so it remains a matter of debate to decide whether they are really relevant for the dynamics or not.

In the context of 2D turbulence described by the Euler-Poisson system, the quasilinear theory developed by Chavanis yields, instead of Eq. (137),

$$\frac{\partial\bar{\omega}}{\partial t}+\mathbf{u}\cdot\boldsymbol{\nabla}\bar{\omega}=\frac{\epsilon^2}{8}\frac{\partial}{\partial r^\mu}\int d^2\mathbf{r}'\,K^{\mu\nu}\delta(\boldsymbol{\xi}\cdot\mathbf{v})$$

$$\times\left\{\bar{\omega}'(\sigma_0-\bar{\omega}')\frac{\partial\bar{\omega}}{\partial r^\mu}-\bar{\omega}(\sigma_0-\bar{\omega})\frac{\partial\bar{\omega}'}{\partial r'^\mu}\right\},$$
$$(148)$$

where $K^{\mu\nu}$ is defined by Eq. (138) and $\bar{\omega}=\bar{\omega}(\mathbf{r},t)$, $\bar{\omega}'=\bar{\omega}(\mathbf{r}',t)$. In addition to the previous conservation laws, this equation guarantees that the coarse-grained vorticity $\bar{\omega}$ remains bounded by the maximum value of the initial distribution, i.e., $\bar{\omega}\leq\sigma_0$. This equation satisfies a H theorem for the Fermi-Dirac entropy introduced by Miller-Robert-Sommeria at equilibrium [26,27]. Our approach provides, therefore, another way of justifying their results from a dynamical point of view. Equation (148) is written for a single level of vorticity σ_0, but it is possible to extend the quasilinear theory to an arbitrary distribution of vorticity levels (in preparation). Our equations should provide, therefore, an interesting and useful parametrization of the 2D Euler equation. It should be recalled in that respect that the usual turbulent diffusion $\nu\Delta\bar{\omega}$ introduced *ad hoc* on the right-hand side of the 2D

Euler equation in order to smooth out the small scales and prevent numerical instabilities breaks the conservation laws of the inviscid dynamics. This is *not* the case for our equation (148): not only does it smooth out the unresolved scales (as exemplified by the existence of a H theorem), but it also satisfies all the conservation laws of the inviscid dynamics and respects the invariance properties of the Euler equation (invariance by translation and rotation of the coordinates, Galilean invariance, and invariance by rotation of the referential). In addition, there is no free parameter in our theory except the coarse-graining mesh ϵ (or resolution scale), which depends on the situation contemplated. Different attempts had been made previously to obtain an equation satisfying all these requirements, but only partial results were obtained [20,9,22,5,23].

C. Connection with the Fokker-Planck equation

Equation (137) can be considered as our final result, but we wish to show that a direct connection with the Fokker-Planck equation of Sec. IV can be found. Introducing a diffusion tensor

$$D^{\mu\nu}=\frac{N\gamma^2}{8}\int d^2\mathbf{r}_1K^{\mu\nu}\delta(\boldsymbol{\xi}\cdot\mathbf{v})P_1\qquad(149)$$

and a drift term

$$\eta^\mu=-\frac{N\gamma^2}{8}\int d^2\mathbf{r}_1K^{\mu\nu}\delta(\boldsymbol{\xi}\cdot\mathbf{v})\frac{\partial P_1}{\partial r_1^\nu},\qquad(150)$$

Eq. (137) can be rewritten in the more illuminating form:

$$\frac{\partial P}{\partial t}+\langle\mathbf{V}\rangle\boldsymbol{\nabla}P=\frac{\partial}{\partial r^\mu}\left[D^{\mu\nu}\frac{\partial P}{\partial r^\nu}+P\eta^\mu\right],\qquad(151)$$

similar to the general Fokker-Planck equation (40). Note, however, that Eq. (151) is an integrodifferential equation since the density probability $P(\mathbf{r},t)$ in $\mathbf{r}$ at time t depends on the value of the whole distribution of probability $P(\mathbf{r}_1,t)$ at the same time by an integration over $\mathbf{r}_1$. In contrast, the Fokker-Planck equation (47) is a differential equation. The usual way to transform an integrodifferential equation into a differential equation is to make a guess for the function $P(\mathbf{r}_1)$ appearing under the integral sign and refine the guess by successive iterations. In practice, we simply make one sensible guess. Therefore, if we are close to equilibrium, it seems natural to replace the function P_1 appearing in the integrals by the Boltzmann distribution,

$$P(\mathbf{r}_1)=Ae^{-\beta\gamma\psi'(\mathbf{r}_1)}.\qquad(152)$$

This corresponds to the "thermal bath approximation" of Sec. IV D: the vortices have not yet relaxed completely, but when we focus on the relaxation of a given point vortex (described by P), we can consider, in a first approximation, that the rest of the system (described by P_1) is at equilibrium. Within this approximation, the diffusion coefficient and the drift simplify in

$$\eta^\mu = \beta\gamma D^{\mu\nu}\frac{\partial\psi'}{\partial r^\nu}, \tag{153}$$

$$D^{\mu\nu} = \frac{N\gamma^2}{8}P(\mathbf{r},t)\int K^{\mu\nu}\delta(\boldsymbol{\xi}\cdot\mathbf{v})d^2\boldsymbol{\xi}, \tag{154}$$

where we have made the local approximation. If we assume that the correlation time is short, i.e., if we replace $\xi^2\delta(\boldsymbol{\xi}\cdot\mathbf{v})$ by τ/π^2 [compare Eqs. (137) and (130)], we obtain

$$\boldsymbol{\eta} = \beta\gamma D\boldsymbol{\nabla}\psi', \tag{155}$$

$$D = \frac{\gamma\tau}{16\pi}\ln N\langle\omega\rangle. \tag{156}$$

In that case, Eq. (151) reduces to the Fokker-Planck equation found in Sec. IV:

$$\frac{\partial P}{\partial t} + \langle\mathbf{V}\rangle\boldsymbol{\nabla}P = \boldsymbol{\nabla}[D(\boldsymbol{\nabla}P + \beta\gamma P\boldsymbol{\nabla}\psi')]. \tag{157}$$

If instead of the Boltzmann distribution (152) we use the Gaussian distribution (132), we get

$$\frac{\partial P}{\partial t} + \langle\mathbf{V}\rangle\boldsymbol{\nabla}P = \boldsymbol{\nabla}[D(\boldsymbol{\nabla}P + \alpha\gamma P\mathbf{r})]. \tag{158}$$

This equation is closely related to the Kramers-Chandrasekhar equation (49) since the drift and the friction are linear in $\mathbf{r}$ and $\mathbf{v}$, respectively.

The diffusion coefficient (156) was previously obtained by [9,28,23] using phenomenological arguments. However, in these studies the correlation time τ was not specified. A first determination of τ was obtained in Ref. [5], but it was restricted to axisymmetrical or unidirectional flows. Using Eq. (154), we can determine the expression of the diffusion coefficient and the correlation time in the general case. Expanding the velocity difference $\mathbf{v} = \langle\mathbf{V}_1\rangle - \langle\mathbf{V}\rangle$ in a Taylor series in $\boldsymbol{\xi} = \mathbf{r}_1 - \mathbf{r}$, we obtain to first order in the expansion

$$\boldsymbol{\xi}\cdot\mathbf{v} = \Sigma^{\mu\nu}\xi^\mu\xi^\nu, \tag{159}$$

where

$$\Sigma^{\mu\nu} = \frac{1}{2}\left(\frac{\partial\langle V\rangle^\mu}{\partial r^\nu} + \frac{\partial\langle V\rangle^\nu}{\partial r^\mu}\right) \tag{160}$$

is the stress tensor. It satisfies the property of symmetry $\Sigma^{\mu\nu} = \Sigma^{\nu\mu}$. Since the flow is divergenceless, we also have $\Sigma^{xx} + \Sigma^{yy} = 0$. This suggests introducing the notations $a = \Sigma^{xx} = -\Sigma^{yy}$ and $b = \Sigma^{xy} = \Sigma^{yx}$. In terms of the stress tensor (160), the diffusion tensor (154) can be rewritten

$$D^{\mu\nu} = \frac{N\gamma^2}{8}P\int\frac{\xi^2\delta^{\mu\nu} - \xi^\mu\xi^\nu}{\xi^2}\delta(\Sigma^{\mu\nu}\xi^\mu\xi^\nu)d^2\boldsymbol{\xi}. \tag{161}$$

This integral can be performed easily by working in a basis where the tensor $\Sigma^{\mu\nu}$ is antidiagonal. For that purpose, we

seek a tensor T such that $\Sigma = \tilde{T}\Sigma'T$, where Σ' is antidiagonal and, by definition, $\tilde{T}^{\mu\nu} = T^{\nu\mu}$. We also impose that T is unitary so that $\tilde{T} = T^{-1}$. Then, if we denote by $(M^{11}, M^{12}, M^{21}, M^{22})$ the components of a 2×2 matrix, we find $\Sigma' = (0, b', b', 0)$ and $T = (\alpha, \beta, -\beta, \alpha)$ with $\alpha^2 + \beta^2 = 1$ and $b' = b/(\alpha^2 - \beta^2) = -a/2\alpha\beta$. From the above results, it is also clear that $b'^2 = -\det(\Sigma) = a^2 + b^2$, where $\det(\Sigma)$ stands for the determinant of the matrix Σ. Now, introducing a new system of coordinates such that $\xi'^\mu = T^{\mu\nu}\xi^\nu$, or alternatively $\xi^\mu = T^{\nu\mu}\xi'^\nu$, we easily check that the Jacobian of the transformation $\boldsymbol{\xi}\to\boldsymbol{\xi}'$, i.e., the determinant of T, is equal to 1. Under these circumstances, the diffusion tensor (161) can be written $D = \tilde{T}D'T$ with

$$D'^{\mu\nu} = \frac{N\gamma^2}{8}P\int\frac{\xi'^2\delta^{\mu\nu} - \xi'^\mu\xi'^\nu}{\xi'^2}\delta(|\Sigma(\mathbf{r})|\xi_1'\xi_2')d\xi_1'd\xi_2', \tag{162}$$

where we have set $|\Sigma(\mathbf{r})| = 2\sqrt{-\det(\Sigma)}$. Physically, this quantity represents the local shear of the flow. The components of the tensor (162) can now be determined easily. First of all,

$$D'^{11} = \frac{N\gamma^2}{8}\frac{1}{|\Sigma(\mathbf{r})|}P\int\frac{\xi_2'^2}{\xi_1'^2 + \xi_2'^2}\delta(\xi_1'\xi_2')\ d\xi_1'd\xi_2'. \tag{163}$$

Setting $\xi_1' = \xi\cos\theta$ and $\xi_2' = \xi\sin\theta$, where $\xi = \xi' = |\mathbf{r}_1 - \mathbf{r}|$, we get

$$D'^{11} = \frac{N\gamma^2}{8}\frac{1}{|\Sigma(\mathbf{r})|}P\int_0^{+\infty}\xi d\xi\int_0^{2\pi}d\theta\sin^2\theta\delta(\xi^2\cos\theta\sin\theta), \tag{164}$$

or, equivalently,

$$D'^{11} = \frac{N\gamma^2}{4}\frac{1}{|\Sigma(\mathbf{r})|}P\int_0^{+\infty}\frac{d\xi}{\xi}\int_0^\pi d\theta\sin\theta\delta(\cos\theta). \tag{165}$$

As explained in Appendix A, we regularize the logarithmic divergence by introducing appropriate cutoffs at small and large scales. With the change of variables $t = \cos\theta$, we obtain

$$D'^{11} = \frac{N\gamma^2}{8}\frac{1}{|\Sigma(\mathbf{r})|}P\ln N\int_{-1}^{+1}dt\,\delta(t) \to \frac{N\gamma^2}{8}\frac{1}{|\Sigma(\mathbf{r})|}P\ln N. \tag{166}$$

By the same arguments, we find that $D'^{22} = D'^{11} = D$. On the other hand, it follows, for reasons of antisymmetry by the transformation $\xi_1' \to -\xi_1'$, that $D'^{12} = D'^{21} = 0$. Therefore, $D'^{\mu\nu} = D\delta^{\mu\nu}$ is diagonal in the basis where Σ is antidiagonal. This remains true in any basis since $D^{\mu\nu} = T^{\lambda\mu}D'^{\lambda\sigma}T^{\sigma\nu} = D(\tilde{T}T)^{\mu\nu} = D\delta^{\mu\nu}$. Therefore, close to equilibrium, the diffusion is isotropic and the general expression of the diffusion coefficient is

$$D = \frac{\gamma}{8} \frac{1}{|\Sigma(\mathbf{r})|} \ln N \langle \omega \rangle. \tag{167}$$

Comparing with Eq. (156), we find that the correlation time is given by

$$\tau = \frac{2\pi}{|\Sigma(\mathbf{r})|}. \tag{168}$$

The quantity $|\Sigma(\mathbf{r})| = 2\sqrt{-\det(\Sigma)}$ plays a fundamental role in the theory. Clearly, this expression is invariant by a change of referential. For a unidirectional flow or an axisymmetrical flow, we recover the results of Sec IV E and of Ref. [5].

VI. CONCLUSION

In this paper, we have provided a systematic derivation of the kinetic equations of point vortices, applying the powerful projection operator techniques to this problem. We have described how a cloud of point vortices relaxes toward the mean-field statistical equilibrium, leading to a clustering into large coherent vortices. In the first part of the paper, we have focused on the relaxation of a "test" vortex in a cloud of background vortices at statistical equilibrium. The cloud of "field" vortices plays the role of a thermal bath, as in other problems of statistical physics. We have shown that the test vortex undergoes a usual diffusion effect due to random fluctuations and that it also experiences a *systematic drift*. This drift, due to a polarization of the background vortices by the test vortex, balances the effect of diffusion at equilibrium, providing a dynamical explanation for the persistence of clustering. The drift was previously derived with a linear-response theory [5], but the diffusion was heuristically introduced by adding a white-noise effect. The present derivation systematically derives the two effects—diffusion and drift—from the same formalism, and is therefore more satisfactory. The diffusion derived here turns out to be influenced by long-time correlations, so it is more complex than the usual white-noise effect. When memory effects are ignored, we obtain a Fokker-Planck equation for the evolution of the one-particle distribution function. This Fokker-Planck equation can also be derived from a phenomenological maximum entropy production principle [20]. This shows that the structure of this equation is influenced more by thermodynamics (the first and second principles) than by the precise microscopic model. However, our systematic procedure starting directly from the Liouville equation provides a *justification* for this thermodynamic approach and specifies its range of validity. It also allows us to determine explicitly the value of the diffusion coefficient, which was left unspecified by the maximum entropy production principle. All these results could be tested numerically by introducing a test vortex in a "sea" of vortices at statistical equilibrium and by solving the Kirchhoff-Hamilton equations of motion.

In the second part of the paper, we have attempted to describe the evolution of the whole system of vortices far from equilibrium. We have obtained a new kinetic equation (128), which incorporates a delocalization in space and time. This is therefore a non-Markovian intregro-differential equa-

tion. A similar equation also occurs in the quasilinear theory of the 2D Euler equation [23]. Within some approximation, it is possible to carry out the time integration explicitly, and this yields a simpler equation (137), which only conserves a delocalization in space. This equation respects all the conservation laws of the point vortex system and satisfies a H theorem. The relaxation is due to a condition of resonance between distant vortices. If the system is sufficiently "resonant," it will reach a maximum entropy state described by the Boltzmann distribution. However, if there are not enough resonances the evolution may stop on a metastable state. Only nontrivial correlations between vortices (for example, three-body collisions) can unfreeze the system and induce further evolution. These correlations are not taken into account in the present analysis, although the projection operator formalism might still apply. It would be necessary to modify the factorization hypothesis (126) so as to account for two-body or three-body correlation functions. It is possible, however, that these correlations develop on a much longer time scale, so it is not yet clear whether they are physically relevant. In any case, the approximations made in the present paper are a first step towards a rational kinetic theory of point vortices.

It is also possible that a system of point vortices undergoes a form of "violent relaxation" in its early stage. For short time scales, the correlations between point vortices have not yet developed and the average vorticity is a solution of the 2D Euler equation. When the initial condition is far from equilibrium, it is well known that the 2D Euler equation develops a complicated mixing process leading to the formation of an organized state (on a coarse-grained scale). This relaxation is quite rapid, of the order of the dynamical time t_D, and the resulting equilibrium state is predicted to be a complicated superposition of Fermi-Dirac distributions respecting all the constraints of the Euler equation [26,27]. On longer time scales, of the order of $(N/\ln N)t_D$, the correlations between point vortices develop and the system undergoes another form of relaxation, much slower. This relaxation is towards the Boltzmann distribution derived by Joyce and Montgomery [3] and Lundgren and Pointin [4], which is the true equilibrium state for a system of point vortices. The first type of relaxation, by vorticity mixing, was described in Ref. [23] using a quasilinear theory of the 2D Euler equation. The exclusion principle leading to the Fermi-Dirac statistics was explicitly shown as well as a *H* theorem. The second type of relaxation, due to discrete interactions between point vortices, was the object of the present paper.

It is noteworthy that a similar distinction occurs in the context of stellar systems [9]. Indeed, the relaxation of stars is a two-stage process. For short time scales $\sim t_D$, the encounters between stars can be neglected and the distribution function is a solution of the Vlasov equation (analogous to the 2D Euler equation). If the system is initially far from mechanical equilibrium, it will experience a "violent relaxation" towards a virialized state. This equilibrium is predicted to be a superposition of Fermi-Dirac statistics [25,29], as for the 2D Euler equation. Then, on a longer time scale, of the order of $(N/\ln N)t_D$, the encounters between stars cannot be ignored anymore and will cause the stars to deviate from

their unperturbed trajectories. This collisional relaxation is usually described by a Landau or Fokker-Planck equation that converges towards the Maxwell-Boltzmann distribution at equilibrium. On even longer time scales, three-body encounters leading to the formation of binaries induce a slow evolution of the system [30]. Three-body encounters (involving vortices of different sign) are also relevant in two-dimensional turbulence and lead to the formation of vortex "pairs" [19]. The analogy with "binary stars" is interesting to note. It would be important to test these ideas with numerical simulations of point vortices or stars. The situation is difficult in the stellar context because a maximum entropy state does not always exist. Indeed, the system can collapse and overheat: this is the so-called "gravothermal catastrophe" [31]. This problem does not occur for point vortices, and it should be possible to show the two (or more) successive equilibria more properly. An advantage of point vortices with respect to stars is the lower dimensionality of space ($D=2$ instead of $D=3$, or $D=6$ in phase space) which should make numerical simulations easier.

ACKNOWLEDGMENTS

APPENDIX A: THE STATISTICS OF VELOCITY FLUCTUATIONS IN AN INHOMOGENEOUS MEDIUM

In this appendix, we study the statistics of velocity fluctuations produced by an inhomogeneous distribution of point vortices. This study extends the calculations of [32,33,13] for a uniform medium and provides a simple framework to understand the logarithmic divergence of the diffusion coefficient. Let us consider a collection of N point vortices randomly distributed in a disk of radius R with an average density $n(\mathbf{r})$. The velocity $\mathbf{V}$ occurring at a given location $\mathbf{r}$ of the flow is the sum of the velocities $\mathbf{\Phi}_i$ ($i=1,\ldots,N$) produced by the N vortices:

$$\mathbf{V}=\sum_{i=1}^{N}\mathbf{\Phi}_i, \tag{A1}$$

$$\mathbf{\Phi}_i=-\frac{\gamma}{2\pi}\frac{(\mathbf{r}_i-\mathbf{r})_{\perp}}{|\mathbf{r}_i-\mathbf{r}|^2}. \tag{A2}$$

Following a procedure similar to that adopted in Ref. [13], the velocity distribution can be expressed as

$$W_N(\mathbf{V})=\frac{1}{4\pi^2}\int A_N(\boldsymbol{\rho})e^{-i\boldsymbol{\rho}\cdot\mathbf{V}}d^2\boldsymbol{\rho} \tag{A3}$$

with

$$A_N(\boldsymbol{\rho})=\left(\int e^{i\boldsymbol{\rho}\cdot\mathbf{\Phi}}P(\mathbf{r}_1)d^2\mathbf{r}_1\right)^{N}. \tag{A4}$$

Here, $P(\mathbf{r}_1)$ denotes the probability of occurrence of a point vortex in $\mathbf{r}_1$ and by definition

$$\mathbf{\Phi}=-\frac{\gamma}{2\pi}\frac{(\mathbf{r}_1-\mathbf{r})_{\perp}}{|\mathbf{r}_1-\mathbf{r}|^2}. \tag{A5}$$

Introducing explicitly the vortex density $n(\mathbf{r}_1)=NP(\mathbf{r}_1)$, we have

$$A_N(\boldsymbol{\rho})=\left(\frac{1}{N}\int n(\mathbf{r}_1)e^{i\boldsymbol{\rho}\cdot\mathbf{\Phi}}d^2\mathbf{r}_1\right)^{N}. \tag{A6}$$

Since

$$\int n(\mathbf{r}_1)d^2\mathbf{r}_1=N, \tag{A7}$$

we can rewrite our expression for $A_N(\boldsymbol{\rho})$ in the form

$$A_N(\boldsymbol{\rho})=\left(1-\frac{1}{N}\int n(\mathbf{r}_1)(1-e^{i\boldsymbol{\rho}\cdot\mathbf{\Phi}})d^2\mathbf{r}_1\right)^{N}. \tag{A8}$$

In the limit of large N,R with fixed $n(\mathbf{r})$, we can approximate the foregoing expression by

$$A(\boldsymbol{\rho})=e^{-C(\boldsymbol{\rho})} \tag{A9}$$

with

$$C(\boldsymbol{\rho})=\int n(\mathbf{r}_1)(1-e^{i\boldsymbol{\rho}\cdot\mathbf{\Phi}})d^2\mathbf{r}_1. \tag{A10}$$

Separating the real and imaginary parts of $C(\boldsymbol{\rho})$, we obtain

$$\begin{aligned}C(\boldsymbol{\rho})&=C_1(\boldsymbol{\rho})-iC_2(\boldsymbol{\rho})\\&=\int n(\mathbf{r}_1)[1-\cos(\boldsymbol{\rho}\cdot\mathbf{\Phi})]d^2\mathbf{r}_1\\&\quad-i\int n(\mathbf{r}_1)\sin(\boldsymbol{\rho}\cdot\mathbf{\Phi})d^2\mathbf{r}_1.\end{aligned} \tag{A11}$$

In the first integral, we find it convenient to introduce the relative separation $\boldsymbol{\xi}=\mathbf{r}_1-\mathbf{r}$ in terms of which

$$C_1(\boldsymbol{\rho})=\int n(\mathbf{r}+\boldsymbol{\xi})[1-\cos(\boldsymbol{\rho}\cdot\mathbf{\Phi})]d^2\boldsymbol{\xi} \tag{A12}$$

with

$$\mathbf{\Phi}=-\frac{\gamma}{2\pi}\frac{\boldsymbol{\xi}_{\perp}}{\xi^2}. \tag{A13}$$

In Ref. [13], it was found that an important contribution to the velocity fluctuations comes from the nearest neigbor. This justifies to making the local approximation $n(\mathbf{r}+\boldsymbol{\xi})\simeq n(\mathbf{r})$ in Eq. (A12). In this approximation,

$$C_1(\boldsymbol{\rho})=n(\mathbf{r})\int[1-\cos(\boldsymbol{\rho}\cdot\mathbf{\Phi})]d^2\boldsymbol{\xi}. \tag{A14}$$

This quantity is closely related to the function $C(\boldsymbol{\rho})$ evaluated in Ref. [13] for a distribution of vortices with uniform density n. The only difference is the presence of the local density $n(\mathbf{r})$ in place of n. Therefore, we can infer directly that

$$C_1(\boldsymbol{\rho}) = \frac{n(\mathbf{r})\gamma^2}{8\pi} \ln\left(\frac{2\pi R}{\gamma\rho}\right)\rho^2. \tag{A15}$$

In fact, the local approximation is only marginally valid because, as discussed in Ref. [13], the contribution of the nearest neighbor is precisely of the same order of magnitude as the contribution of the rest of the system. This results in a logarithmic divergence in Eq. (A15) due to the weak collective behavior of the system.

In the second integral appearing in Eq. (A11), the contribution from proximate vortices vanishes by symmetry. As a result, the integral is dominated by large values of $|\mathbf{r}_1|$ or, equivalently, by small values of $|\boldsymbol{\Phi}|$. We can therefore make the approximation $\sin(\boldsymbol{\rho}\cdot\boldsymbol{\Phi})\simeq\boldsymbol{\rho}\cdot\boldsymbol{\Phi}$ and write

$$C_2(\boldsymbol{\rho}) = \boldsymbol{\rho}\int n(\mathbf{r}_1)\boldsymbol{\Phi}d^2\mathbf{r}_1. \tag{A16}$$

In the integral, we recognize the mean-field velocity created in $\mathbf{r}$ by the average distribution of vortices:

$$\langle\mathbf{V}\rangle(\mathbf{r}) = \int n(\mathbf{r}_1)\boldsymbol{\Phi}d^2\mathbf{r}_1. \tag{A17}$$

Hence,

$$C_2(\boldsymbol{\rho}) = \boldsymbol{\rho}\langle\mathbf{V}\rangle(\mathbf{r}). \tag{A18}$$

Substituting the explicit expression for $C(\boldsymbol{\rho})$ in Eq. (A9), we obtain

$$A(\boldsymbol{\rho}) = e^{-[n(\mathbf{r})\gamma^2/8\pi]\,\ln(2\pi R/\gamma\rho)\rho^2 + i\boldsymbol{\rho}\langle\mathbf{V}\rangle(\mathbf{r})}. \tag{A19}$$

Therefore, the velocity distribution in $\mathbf{r}$ can be written quite generally,

$$W(\mathbf{V}) = \frac{1}{4\pi^2}\int e^{-[n(\mathbf{r})\gamma^2/8\pi]\,\ln(2\pi R/\gamma\rho)\rho^2}e^{-i\boldsymbol{\rho}[\mathbf{V}-\langle\mathbf{V}\rangle(\mathbf{r})]}d^2\boldsymbol{\rho}. \tag{A20}$$

This is the same distribution as for a uniform distribution of vortices except that the constant density n has been replaced by the local density $n(\mathbf{r})$ and the distribution is for the fluctuating velocity $\mathcal{V} = \mathbf{V} - \langle\mathbf{V}\rangle(\mathbf{r})$. Repeating the calculations of Ref. [13], the velocity p.d.f is explicitly given by

$$W(\mathcal{V}) = \frac{4}{n(\mathbf{r})\gamma^2\ln N}e^{-4\pi/[n(\mathbf{r})\gamma^2\ln N]\mathcal{V}^2} \quad [\mathcal{V}\lesssim\mathcal{V}_{\mathrm{crit}}(N)], \tag{A21}$$

$$W(\mathcal{V}) \sim \frac{n(\mathbf{r})\gamma^2}{4\pi^2\mathcal{V}^4} \quad [\mathcal{V}\gtrsim\mathcal{V}_{\mathrm{crit}}(N)], \tag{A22}$$

where

$$\mathcal{V}_{\mathrm{crit}}(N) \sim \left(\frac{n(\mathbf{r})\gamma^2}{4\pi}\ln N\right)^{1/2}\ln^{1/2}(\ln N). \tag{A23}$$

This distribution lies at the frontier between Gaussian and Lévy laws: the core of the distribution is Gaussian while the tail decays algebraically as for Lévy laws. This is because the variance of the individual velocities (A2) diverges logarithmically so the central-limit theorem is only marginally applicable. For that reason, we have proposed to call this distribution the "marginal Gaussian distribution" [14]. In the strict mathematical limit $N\to+\infty$, the transition between the two regimes is rejected to infinity and the velocity p.d.f. is purely Gaussian [33]. However, the convergence towards this Gaussian distribution is so slow that in practical applications it is never reached: the algebraic tail always remain [18].

According to Eqs. (A1) and (A2), the variance of the velocity can be written

$$\langle\mathcal{V}^2\rangle = \int_{|\boldsymbol{\xi}|=0}^{R} n(\mathbf{r}+\boldsymbol{\xi})\frac{\gamma^2}{4\pi^2\xi^2}d^2\boldsymbol{\xi} = n(\mathbf{r})\int_0^R\frac{\gamma^2}{4\pi^2\xi^2}2\pi\xi d\xi, \tag{A24}$$

where we have made the local approximation in the second equality. This quantity diverges logarithmically at both small and large vortex separations. The divergence at small separations is a failure of our model, which ignores correlations between vortices. In fact, when two vortices approach each other, they can form a pair, as discussed in Ref. [18], and our mean-field theory clearly breaks down. We shall account heuristically for this failure by introducing a cutoff at some minimum distance $d_{\mathrm{pair}}\sim(\pi n\ln N)^{-1/2}$ [13]. The divergence at large separations is due to the unshielded nature of the interaction potential. It is therefore natural to cut the integral at R, the typical size of the system. With this regularization, we obtain

$$\langle\mathcal{V}^2\rangle = \frac{n(\mathbf{r})\gamma^2}{2\pi}\ln\left(\frac{R}{d}\right) = \frac{n(\mathbf{r})\gamma^2}{4\pi}\ln N. \tag{A25}$$

Since the divergence is weak (logarithmic), the result does not depend crucially on the precise value of the cutoffs. The same expression for the variance can also be obtained from the formula

$$\langle\mathcal{V}^2\rangle = \int W(\mathcal{V})\mathcal{V}^2 2\pi\mathcal{V}d\mathcal{V} \tag{A26}$$

if we introduce a cutoff at large velocities and use Eqs. (A21), (A22), and (A23) [to sufficient accuracy, we just need to consider Eq. (A21)].

The diffusion coefficient can be expressed in terms of the variance (A25) and the typical correlation time τ by [13]

$$D = \tfrac{1}{4}\tau\langle\mathcal{V}^2\rangle. \tag{A27}$$

Using Eq. (A25) and the relation $\langle\omega\rangle = n\gamma$, we obtain

$$D = \frac{\gamma\tau}{16\pi}\ln N\langle\omega\rangle \tag{A28}$$

in agreement with our result (156). The correlation time is more difficult to evaluate. If we ignore the differential rotation of the vortices, the calculations of Ref. [13] are directly applicable and lead to the expression

$$\tau \sim \frac{1}{\langle \omega \rangle (\mathbf{r}) \sqrt{\ln N}}. \tag{A29}$$

This is the typical time needed by a vortex to cross the interparticle distance $d \sim 1/\sqrt{n(\mathbf{r})}$ with the velocity $\sqrt{\langle \mathcal{V}^2 \rangle}$. With this approximation, the diffusion coefficient is given by Eq. (121). On the other hand, if we consider that the vortices are transported by the equilibrium flow and evaluate the diffusion coefficient with the Kubo formula (see Appendix B), we find that the correlation time is related to the local shear by Eq. (168), i.e.,

$$\tau = \frac{2\pi}{|\Sigma(\mathbf{r})|}. \tag{A30}$$

Physically, it corresponds to the time needed by two vortices with relative velocity Σd to be stretched by the shear on a distance $\sim d$. This approximation breaks down, however, when the shear is weak. In that case, τ is given by Eq. (A29) obtained when only the dispersion of the vortices is considered. Clearly, a general formula should take into account simultaneously the effect of the shear and the dispersion of the particles.

APPENDIX B: THE KUBO FORMULA

Let us consider the diffusion of a test vortex in a ''sea'' of field vortices described by the equilibrium distribution

$$\mu_{eq}(\mathbf{r}_1, \ldots, \mathbf{r}_N) = \prod_{i=1}^{N} P_{eq}(\mathbf{r}_i). \tag{B1}$$

The general form of the diffusion coefficient, as defined by Eq. (40), reads

$$D^{\mu\nu} = \frac{\langle \Delta r^{\mu} \Delta r^{\nu} \rangle}{2\Delta t}. \tag{B2}$$

Now, the net displacement of the test vortex produced by the fluctuations of the velocity between t and $t + \Delta t$ is given by

$$\Delta \mathbf{r} = \int_{t}^{t+\Delta t} \mathcal{V}(t') dt'. \tag{B3}$$

Substituting explicitly for $\Delta \mathbf{r}$ from Eq. (B3) in Eq. (B2), we have

$$D^{\mu\nu} = \frac{1}{2\Delta t} \int_{0}^{\Delta t} dt' \int_{0}^{\Delta t} dt'' \langle \mathcal{V}^{\mu}(t+t') \mathcal{V}^{\nu}(t+t'') \rangle_{eq}, \tag{B4}$$

where $\langle \rangle_{eq}$ denotes the average with respect to the equilibrium distribution (B1). Since the correlation function appearing in the integral only depends on the time difference $|t''-t'|$, we also have

$$D^{\mu\nu} = \frac{1}{\Delta t} \int_{0}^{\Delta t} dt' \int_{0}^{t'} dt'' \langle \mathcal{V}^{\mu}(t+t') \mathcal{V}^{\nu}(t+t'') \rangle_{eq} \tag{B5}$$

or, alternatively,

$$D^{\mu\nu} = \frac{1}{\Delta t} \int_{0}^{\Delta t} dt' \int_{0}^{t'} dt'' \langle \mathcal{V}^{\mu}(t) \mathcal{V}^{\nu}(t+t''-t') \rangle_{eq}. \tag{B6}$$

With the change of variables $\tau = t' - t''$, we obtain successively

$$\begin{aligned} D^{\mu\nu} &= \frac{1}{\Delta t} \int_{0}^{\Delta t} dt' \int_{0}^{t'} d\tau \langle \mathcal{V}^{\mu}(t) \mathcal{V}^{\nu}(t-\tau) \rangle_{eq} \\ &= \frac{1}{\Delta t} \int_{0}^{\Delta t} d\tau \int_{\tau}^{\Delta t} dt' \langle \mathcal{V}^{\mu}(t) \mathcal{V}^{\nu}(t-\tau) \rangle_{eq} \\ &= \frac{1}{\Delta t} \int_{0}^{\Delta t} d\tau \langle \mathcal{V}^{\mu}(t) \mathcal{V}^{\nu}(t-\tau) \rangle_{eq} (\Delta t - \tau). \end{aligned} \tag{B7}$$

If the correlation function $\langle \mathcal{V}^{\mu}(t) \mathcal{V}^{\nu}(t-\tau) \rangle_{eq}$ decays more rapidly than τ^{-1}, we can take the limit $\Delta t \to +\infty$ to finally obtain

$$D^{\mu\nu} = \int_{0}^{+\infty} \langle \mathcal{V}^{\mu}(t) \mathcal{V}^{\nu}(t-\tau) \rangle_{eq} d\tau. \tag{B8}$$

This is the Kubo formula for our problem. Remembering that $\mathcal{V}$ denotes the fluctuation of the total velocity,

$$\mathcal{V}(t) = \mathbf{V}(t) - \langle \mathbf{V}(t) \rangle_{eq}, \tag{B9}$$

we find that

$$\begin{aligned} \langle \mathcal{V}^{\mu}(t) \mathcal{V}^{\nu}(t-\tau) \rangle_{eq} &= \langle V^{\mu}(t) V^{\nu}(t-\tau) \rangle_{eq} \\ &\quad - \langle V^{\mu}(t) \rangle_{eq} \langle V^{\nu}(t-\tau) \rangle_{eq}. \end{aligned} \tag{B10}$$

Now, the first quantity in brackets can be written explicitly as

$$\begin{aligned} &\langle V^{\mu}(t) V^{\nu}(t-\tau) \rangle_{eq} \\ &= \sum_{i=1}^{N} \sum_{j=1}^{N} \int V^{\mu}(i \to 0, t) V^{\nu}(j \to 0, t-\tau) \mu_{eq}(\{\mathbf{r}_k\}) \\ &\quad \times \prod_{k=1}^{N} d^2 \mathbf{r}_k \\ &= \sum_{i=1}^{N} \sum_{j \neq i} \int V^{\mu}(i \to 0, t) V^{\nu}(j \to 0, t-\tau) \mu_{eq}(\{\mathbf{r}_k\}) \\ &\quad \times \prod_{k=1}^{N} d^2 \mathbf{r}_k + \sum_{i=1}^{N} \int V^{\mu}(i \to 0, t) V^{\nu}(i \to 0, t-\tau) \end{aligned}$$

$$\mu_{\mathrm{eq}}(\{\mathbf{r}_k\})\prod_{k=1}^{N}d^2\mathbf{r}_k$$

$$=N(N-1)\langle V^\mu(1\to0,t)\rangle_{\mathrm{eq}}\langle V^\nu(1\to0,t-\tau)\rangle_{\mathrm{eq}}$$

$$+N\int V^\mu(1\to0,t)V^\nu(1\to0,t-\tau)P_{\mathrm{eq}}(\mathbf{r}_1)d^2\mathbf{r}_1.$$

$$(B11)$$

For large N, we can make the approximation $N(N-1)\simeq N^2$. Using Eqs. (B10) and (B11), the correlation function can be set in the form

$$\langle\mathcal{V}^\mu(t)\mathcal{V}^\nu(t-\tau)\rangle_{\mathrm{eq}}=N\int V^\mu(1\to0,t)V^\nu(1\to0,t-\tau)$$

$$\times P_{\mathrm{eq}}(\mathbf{r}_1)d^2\mathbf{r}_1.\quad(B12)$$

Since the integral is dominated by interactions involving relatively close vortices, we can make the local approximation

$$\langle\mathcal{V}^\mu(t)\mathcal{V}^\nu(t-\tau)\rangle_{\mathrm{eq}}=N\int V^\mu(1\to0,t)V^\nu(1\to0,t-\tau)$$

$$\times P_{\mathrm{eq}}(\mathbf{r})d^2\mathbf{r}_1.\quad(B13)$$

The expression for the diffusion coefficient then becomes

$$D^{\mu\nu}=N\int_0^{+\infty}d\tau\int V^\mu(1\to0,t)V^\nu(1\to0,t-\tau)P_{\mathrm{eq}}(\mathbf{r})d^2\mathbf{r}_1.$$

$$(B14)$$

For the sake of brevity, we shall denote the velocity correlation function by

$$C^{\mu\nu}(\tau)\equiv\langle\mathcal{V}^\mu(t)\mathcal{V}^\nu(t-\tau)\rangle_{\mathrm{eq}}$$

$$=N\int V^\mu(1\to0,t)V^\nu(1\to0,t-\tau)P_{\mathrm{eq}}(\mathbf{r})d^2\mathbf{r}_1.$$

$$(B15)$$

Therefore, the Kubo formula takes the form

$$D^{\mu\nu}=\int_0^{+\infty}C^{\mu\nu}(\tau)d\tau.\quad(B16)$$

More generally, we have

$$\langle\Delta r^\mu\Delta r^\nu\rangle=2\int_0^{\Delta t}C^{\mu\nu}(\tau)(\Delta t-\tau)d\tau.\quad(B17)$$

These quantities are calculated explicitly in Appendix C in the case of simple flows.

APPENDIX C: THE CALCULATION OF THE DIFFUSION COEFFICIENT

In this appendix, we calculate the Kubo integral using an approximation in which the point vortices follow the streamlines of the equilibrium flow.

1. Unidirectional flow

We shall first calculate the velocity correlation function (B15) and the diffusion coefficient (B14) in the case of a unidirectional equilibrium flow. The trajectory of a fluid particle advected by this flow is simply.

$$y(t-\tau)=y(t),\quad(C1)$$

$$x(t-\tau)=x(t)-\langle V\rangle_{\mathrm{eq}}(y)\tau.\quad(C2)$$

According to Sec. IV E 1, we are particularly interested by the yy component (103) of the velocity correlation function. Explicitly, it has the form

$$C(\tau)=\frac{N\gamma^2}{4\pi^2}\int dx_1dy_1$$

$$\times\frac{x_1-x}{(x_1-x)^2+(y_1-y)^2}(t)$$

$$\times\frac{x_1-x}{(x_1-x)^2+(y_1-y)^2}(t-\tau)P_{\mathrm{eq}}(y),\quad(C3)$$

where we have used Eq. (7). The second term involves the quantity

$$(x_1-x)(t-\tau)=x_1-x+[\langle V\rangle_{\mathrm{eq}}(y_1)-\langle V\rangle_{\mathrm{eq}}(y)]\tau.$$

$$(C4)$$

In the local approximation, we can expand the velocity difference in a Taylor series in y_1-y. To first order, we have

$$\langle V\rangle_{\mathrm{eq}}(y_1)-\langle V\rangle_{\mathrm{eq}}(y)\simeq-\Sigma(y)(y_1-y),\quad(C5)$$

where $\Sigma(y)$ is the local shear of the flow (105). Introducing the variables $X\equiv x_1-x$ and $Y\equiv y_1-y$, we obtain

$$C(\tau)=\frac{N\gamma^2}{4\pi^2}P_{\mathrm{eq}}(y)\int dXdY\frac{X}{X^2+Y^2}\frac{X+\Sigma(y)Y\tau}{[X+\Sigma(y)Y\tau]^2+Y^2}.$$

$$(C6)$$

The integration over X can be performed easily since the integrand is just a rational function of polynomials. After straightforward calculations, we find

$$C(\tau)=\frac{N\gamma^2}{4\pi}P_{\mathrm{eq}}(y)\frac{1}{1+\frac{1}{4}\Sigma^2(y)\tau^2}\int_0^{+\infty}\frac{dY}{Y}.\quad(C7)$$

The integral over Y diverges logarithmically for both small and large Y. The reason for this divergence has been given in Ref. [13] and in Appendix A. Introducing two cutoffs at scales d and R, and noting that $\ln(R/d)\sim\frac{1}{2}\ln N$, we finally obtain Eq. (104). For $\tau\to+\infty$, the correlation function decreases like τ^{-2}. This is a slow decay but still the diffusion coefficient (B16) converges. Using

$$\int_0^t C(\tau)d\tau = \frac{N\gamma^2}{4\pi}\frac{\ln N}{|\Sigma(y)|}\arctan\left(\frac{1}{2}|\Sigma(y)|t\right)P_{eq}(y) \tag{C8}$$

and taking the limit $t\to +\infty$, we find Eq. (109). More generally, using Eqs. (B17) and (C8), we have

$$\langle(\Delta y)^2\rangle = \frac{N\gamma^2}{2\pi}\frac{\ln N}{|\Sigma(y)|}P_{eq}(y)\left[\arctan\left(\frac{1}{2}|\Sigma(y)|\Delta t\right)\Delta t\right.$$
$$\left. -\frac{1}{|\Sigma(y)|}\ln\left(1+\frac{1}{4}\Sigma(y)^2(\Delta t)^2\right)\right]. \tag{C9}$$

For $\Delta t\to 0$ (ballistic motion),

$$\langle(\Delta y)^2\rangle = \frac{N\gamma^2}{8\pi}\ln N P_{eq}(y)(\Delta t)^2 = \frac{1}{2}\langle\mathcal{V}^2\rangle(\Delta t)^2, \tag{C10}$$

and for $\Delta t\to +\infty$ (diffusive motion),

$$\langle(\Delta y)^2\rangle = \frac{N\gamma^2}{4}\frac{\ln N}{|\Sigma(y)|}P_{eq}(y)\Delta t. \tag{C11}$$

2. Axisymmetrical flow

In an axisymmetrical flow, the trajectory of a fluid particle takes the simple form

$$r(t-\tau) = r(t), \tag{C12}$$

$$\theta(t-\tau) = \theta(t) - \frac{\langle V\rangle_{eq}(r)}{r}\tau. \tag{C13}$$

As indicated in Sec. IV E 2, we are particularly interested in the $r(t)r(t-\tau)$ component (112) of the correlation function. Let us introduce the separation $\delta\mathbf{r}\equiv\mathbf{r}_1-\mathbf{r}$ between the field vortex 1 and the test vortex. In the local approximation, $\delta\mathbf{r}$ can be considered as a small quantity. Therefore we can write

$$\delta\mathbf{r} = r\delta\theta\mathbf{e}_\theta + \delta r\mathbf{e}_r \equiv X\mathbf{e}_\theta + Y\mathbf{e}_r, \tag{C14}$$

$$d^2\mathbf{r}_1 = d^2(\delta\mathbf{r}) = dXdY. \tag{C15}$$

With these notations, the correlation function (112) can be rewritten as

$$C(\tau) = \frac{N\gamma^2}{4\pi^2}P_{eq}(r)\int dXdY\frac{X}{X^2+Y^2}(t)\frac{X}{X^2+Y^2}(t-\tau). \tag{C16}$$

Now,

$$Y(t-\tau) = \delta r(t-\tau) = r_1(t-\tau) - r(t-\tau) = r_1(t) - r(t)$$
$$= Y(t) = Y \tag{C17}$$

and

$$X(t-\tau) = r(t-\tau)\delta\theta(t-\tau) = r(t-\tau)[\theta_1(t-\tau) - \theta(t-\tau)]$$
$$= r\left[\theta_1(t) - \theta(t) - \left(\frac{\langle V\rangle_{eq}(r_1)}{r_1} - \frac{\langle V\rangle_{eq}(r)}{r}\right)\tau\right]. \tag{C18}$$

In the local approximation, we can expand the last term in Eq. (C18) in a Taylor series in r_1-r. This yields

$$X(t-\tau) = r[\theta_1(t) - \theta(t)] - r\frac{d}{dr}\left(\frac{\langle V\rangle_{eq}(r)}{r}\right)(r_1-r)\tau$$
$$= X - \Sigma(r)Y\tau, \tag{C19}$$

where $\Sigma(r)$ is the local shear of the flow (114). Substituting Eqs. (C17) and (C19) into Eq. (C16), we get

$$C(\tau) = \frac{N\gamma^2}{4\pi^2}P_{eq}(r)\int dXdY\frac{X}{X^2+Y^2}\frac{X-\Sigma(r)Y\tau}{[X-\Sigma(r)Y\tau]^2+Y^2}. \tag{C20}$$

This integral is similar to Eq. (C6), so we directly obtain Eqs. (113) and (118).

APPENDIX D: CONSERVATION LAWS SATISFIED BY THE GENERALIZED KINETIC EQUATION

In this section, we prove some general properties satisfied by Eq. (128). Note first that it can be written

$$\frac{\partial P}{\partial t} + \langle\mathbf{V}\rangle\mathbf{\nabla}P = -\mathbf{\nabla}\cdot\mathbf{J}, \tag{D1}$$

where

$$\mathbf{J} = -N\int_0^t d\tau\int d^2\mathbf{r}_1\mathbf{V}(1\to 0)_t\{\mathbf{V}(1\to 0)P_1\mathbf{\nabla}P$$
$$+ \mathbf{V}(0\to 1)P\mathbf{\nabla}P_1\}_{t-\tau} \tag{D2}$$

is the diffusion current. It is clear at first sight that Eq. (D1) conserves the total circulation $\Gamma = \int\langle\omega\rangle d^2\mathbf{r}$ provided that $\mathbf{J}\cdot\hat{\mathbf{n}} = 0$ on the boundary. We now prove the conservation of other integral constraints depending on the domain shape.

(i) In a circular and in an infinite domain, the angular momentum defined by

$$L = \int\langle\omega\rangle r^2 d^2\mathbf{r} \tag{D3}$$

must be conserved. Taking the time derivative of Eq. (D3), substituting for Eq. (D1), and remembering that L is conserved by the advective term, we get

$$\dot{L} = 2N\gamma\int\mathbf{J}\cdot\mathbf{r}d^2\mathbf{r}. \tag{D4}$$

Substituting explicitly for the diffusion current (D2) in Eq. (D4), we obtain

$$\dot{L} = -2N^2\gamma \int_0^t d\tau \int d^2\mathbf{r} d^2\mathbf{r}_1 [\mathbf{r}\cdot\mathbf{V}(1\rightarrow0)]_t \{\mathbf{V}(1\rightarrow0)P_1\nabla P$$

$$+ \mathbf{V}(0\rightarrow1)P\nabla P_1\}_{t-\tau}. \tag{D5}$$

Permuting the dummy variables $\mathbf{r}$ and $\mathbf{r}_1$, we get

$$\dot{L} = -2N^2\gamma \int_0^t d\tau \int d^2\mathbf{r} d^2\mathbf{r}_1 [\mathbf{r}_1\cdot\mathbf{V}(0\rightarrow1)]_t$$

$$\times \{\mathbf{V}(1\rightarrow0)P_1\nabla P + \mathbf{V}(0\rightarrow1)P\nabla P_1\}_{t-\tau}. \tag{D6}$$

Adding these two quantities, we arrive at the final expression

$$\dot{L} = -N^2\gamma \int_0^t d\tau \int d^2\mathbf{r} d^2\mathbf{r}_1 [\mathbf{r}\cdot\mathbf{V}(1\rightarrow0)+\mathbf{r}_1\cdot\mathbf{V}(0\rightarrow1)]_t$$

$$\times \{\mathbf{V}(1\rightarrow0)P_1\nabla P + \mathbf{V}(0\rightarrow1)P\nabla P_1\}_{t-\tau}. \tag{D7}$$

Now, the term in curly brackets vanishes as shown by the following argument. Consider two point vortices in a circular (or infinite) domain. Their angular momentum is

$$L = \gamma(r^2 + r_1^2) \tag{D8}$$

and it is conserved. This implies

$$0 = \frac{dL}{dt} = 2\gamma \left(\mathbf{r}\cdot\frac{d\mathbf{r}}{dt} + \mathbf{r}_1\cdot\frac{d\mathbf{r}_1}{dt} \right)$$

$$= 2\gamma [\mathbf{r}\cdot\mathbf{V}(1\rightarrow0)+\mathbf{r}_1\cdot\mathbf{V}(0\rightarrow1)]. \tag{D9}$$

We can also prove this result by a direct calculation. In an unbounded domain $\mathbf{V}(0\rightarrow1) = -\mathbf{V}(1\rightarrow0)$ and consequently

$$\mathbf{r}\cdot\mathbf{V}(1\rightarrow0)+\mathbf{r}_1\cdot\mathbf{V}(0\rightarrow1) = \mathbf{V}(1\rightarrow0)\cdot(\mathbf{r}-\mathbf{r}_1) = 0, \tag{D10}$$

where we have used Eq. (7) to get the last equality. In a circular domain, the velocity $\mathbf{V}(1\rightarrow0)$ is given by Eq. (7) *plus* a term $\mathbf{V}_b(1\rightarrow0)$ which can be determined with the method of "images" if appropriate. If R denotes the domain radius, we find

$$\mathbf{V}_b(1\rightarrow0) = \frac{\gamma}{2\pi}\hat{\mathbf{z}}\times\frac{\dfrac{R^2}{r_1^2}\mathbf{r}_1 - \mathbf{r}}{\left|\dfrac{R^2}{r_1^2}\mathbf{r}_1 - \mathbf{r}\right|^2}. \tag{D11}$$

Therefore,

$$\mathbf{r}\cdot\mathbf{V}_b(1\rightarrow0)+\mathbf{r}_1\cdot\mathbf{V}_b(0\rightarrow1)$$

$$= \frac{\gamma}{2\pi}\left\{ \frac{(\hat{\mathbf{z}}\times\mathbf{r}_1)\cdot\mathbf{r}}{\left|\dfrac{R}{r_1}\mathbf{r}_1 - \dfrac{r_1}{R}\mathbf{r}\right|^2} + \frac{(\hat{\mathbf{z}}\times\mathbf{r})\cdot\mathbf{r}_1}{\left|\dfrac{R}{r}\mathbf{r} - \dfrac{r}{R}\mathbf{r}_1\right|^2} \right\}. \tag{D12}$$

Noting that $(\hat{\mathbf{z}}\times\mathbf{r}_1)\cdot\mathbf{r} = -(\hat{\mathbf{z}}\times\mathbf{r})\cdot\mathbf{r}_1$ and that

$$\left|\frac{R}{r_1}\mathbf{r}_1 - \frac{r_1}{R}\mathbf{r}\right|^2 = R^2 + \frac{r_1^2}{R^2}r^2 - 2\mathbf{r}\cdot\mathbf{r}_1 = \left|\frac{R}{r}\mathbf{r} - \frac{r}{R}\mathbf{r}_1\right|^2, \tag{D13}$$

we finally conclude that

$$\mathbf{r}\cdot\mathbf{V}(1\rightarrow0)+\mathbf{r}_1\cdot\mathbf{V}(0\rightarrow1) = 0. \tag{D14}$$

From this identity and from Eq. (D7), it results that the kinetic equation (128) conserves the angular momentum in a disk and in an infinite domain, i.e., $\dot{L}=0$.

(ii) In an infinite domain or in a channel, the linear impulse

$$\mathbf{P} = \int \mathbf{r}\times\langle\omega\rangle\hat{\mathbf{z}}d^2\mathbf{r} \tag{D15}$$

must be conserved (in a channel extending in the x direction, only the component P_x of the linear impulse must be conserved). Taking the time derivative of Eq. (D15), substituting for Eq. (D1), and remembering that $\mathbf{P}$ is conserved by the advective term, we get

$$\dot{\mathbf{P}}_\perp = N\gamma \int \mathbf{J}d^2\mathbf{r}. \tag{D16}$$

Substituting explicitly for the diffusion current (D2) in Eq. (D16), permuting the dummy variables $\mathbf{r}$ and $\mathbf{r}_1$, and taking the half-sum of the resulting expressions, we finally obtain

$$\dot{\mathbf{P}}_\perp = -\frac{N^2\gamma}{2}\int_0^t d\tau \int d^2\mathbf{r} d^2\mathbf{r}_1 [\mathbf{V}(1\rightarrow0)+\mathbf{V}(0\rightarrow1)]_t$$

$$\times \{\mathbf{V}(1\rightarrow0)P_1\nabla P + \mathbf{V}(0\rightarrow1)P\nabla P_1\}_{t-\tau}. \tag{D17}$$

Now, we can use the same argument as before to show that the term in curly brackets vanishes. Let us consider two point vortices in a channel (or in an infinite domain). Their linear impulse is

$$\mathbf{P}_\perp = \gamma(\mathbf{r}+\mathbf{r}_1) \tag{D18}$$

and it is conserved. This implies

$$\mathbf{0} = \frac{d\mathbf{P}_\perp}{dt} = \gamma\left(\frac{d\mathbf{r}}{dt} + \frac{d\mathbf{r}_1}{dt}\right) = \gamma[\mathbf{V}(1\rightarrow0)+\mathbf{V}(0\rightarrow1)]. \tag{D19}$$

We can also prove this result by a direct calculation. Equation (D19) is obvious in an unbounded domain since $\mathbf{V}(0\rightarrow1) = -\mathbf{V}(1\rightarrow0)$. In a channel extending in the x direction, we need to show that $\dot{P}_x=0$, i.e., $V_y(1\rightarrow0)+V_y(0\rightarrow1) = 0$. Now, the velocity $V_y(1\rightarrow0)$ is given by Eq. (7) *plus* a term $V_b(1\rightarrow0)_y$ which can be determined with the method of "images." If a denotes the width of the channel, we find that

$$V_b(1\to0)_y = -\frac{\gamma}{2\pi} \sum_{n=-\infty}^{+\infty} \left\{ \frac{x-x_1}{(x-x_1)^2+(y-y_1-2na)^2} \right.$$

$$\left. -\frac{x-x_1}{(x-x_1)^2+(y+y_1-2na)^2} \right\}. \qquad (D20)$$

Under this form, it is clear that $V_b(1\to0)_y$ is antisymmetric under the exchange of 1 and 0, so that finally

$$V_y(1\to0)+V_y(0\to1)=0. \qquad (D21)$$

From this identity and from Eq. (D17), it results that the kinetic equation (128) conserves the linear impulse in a channel or in an infinite domain, i.e., $\dot{\mathbf{P}}=\mathbf{0}$.

APPENDIX E: CALCULATION OF THE MEMORY FUNCTION

In this appendix, we calculate the memory function that occurs in Eq. (128).

1. Axisymmetrical flow

If we assume that $P=P(r,t)$, then Eq. (128) simplifies to

$$\frac{\partial P}{\partial t} = -\frac{1}{r}\frac{\partial}{\partial r}(rJ_r), \qquad (E1)$$

where

$$J_r = -N\int_0^t d\tau \int d^2\mathbf{r}_1 V_{r(t)}(1\to0)_t \left\{ V_{r(t-\tau)}(1\to0)P_1\frac{\partial P}{\partial r} \right.$$

$$\left. -V_{r_1(t-\tau)}(1\to0)P\frac{\partial P_1}{\partial r_1} \right\}_{t-\tau} \qquad (E2)$$

and where $V_{r(t)}(1\to0)$ is the component of the vector $\mathbf{V}(1\to0)$ in the direction of $\mathbf{r}(t)$. If we denote by $(r(t),\theta(t))$ and $(r_1(t),\theta_1(t))$ the polar coordinates that specify the position of the point vortices 0 and 1 at time t, we easily find that

$$V_{r(t)}(1\to0) = -\frac{\gamma}{2\pi}\frac{r_1\sin(\theta-\theta_1)}{r_1^2+r^2-2rr_1\cos(\theta-\theta_1)}. \qquad (E3)$$

We shall assume that between t and $t-\tau$, the point vortices follow circular trajectories with angular velocity $\Omega(r,t)$. In that case, $r(t-\tau)=r$ and $\theta(t-\tau)=\theta-\Omega(r,t)\tau$. Then, we obtain

$$V_{r(t-\tau)}(1\to0) = -\frac{\gamma}{2\pi}\frac{r_1\sin(\theta-\theta_1-\Delta\Omega\tau)}{r_1^2+r^2-2rr_1\cos(\theta-\theta_1-\Delta\Omega\tau)} \qquad (E4)$$

with

$$\Delta\Omega = \Omega(r,t)-\Omega(r_1,t). \qquad (E5)$$

We find similarly that $V_{r_1(t-\tau)}(1\to0)=(r/r_1)V_{r(t-\tau)}(1\to0)$. Our previous assumptions also imply that $P(r(t$

$-\tau),t-\tau)\simeq P(r,t)$ between t and $t-\tau$. In other words, this means that the correlation time is smaller than the time scale on which the average vorticity changes appreciably. We do not assume that it is *much* smaller, so this approximation is not over restrictive. In that case, the diffusion current becomes

$$J_r = -N\int_0^{+\infty} d\tau \int_0^{2\pi} d\theta_1 \int_0^{+\infty} rr_1 dr_1 V_{r(t)}(1\to0)$$

$$\times V_{r(t-\tau)}(1\to0)\left[\frac{1}{r}P_1\frac{\partial P}{\partial r}-\frac{1}{r_1}P\frac{\partial P_1}{\partial r_1}\right], \qquad (E6)$$

where the time integral has been extended to $+\infty$. We now need to evaluate the memory function

$$M = \int_0^{+\infty} d\tau \int_0^{2\pi} d\theta_1 V_{r(t)}(1\to0)V_{r(t-\tau)}(1\to0). \qquad (E7)$$

Introducing the notations $\phi=\theta_1-\theta$ and

$$\lambda = \frac{2rr_1}{r_1^2+r^2}<1, \qquad (E8)$$

we have explicitly

$$M = \left(\frac{\gamma\lambda}{4\pi r}\right)^2 \int_0^{+\infty} d\tau \int_0^{2\pi} d\phi \frac{\sin\phi}{1-\lambda\cos\phi}$$

$$\times \frac{\sin(\phi+\Delta\Omega\tau)}{1-\lambda\cos(\phi+\Delta\Omega\tau)}. \qquad (E9)$$

This can also be written

$$M = \left(\frac{\gamma}{4\pi r}\right)^2 \int_0^{+\infty} d\tau \int_0^{2\pi} d\phi V'(\phi)V'(\phi+\Delta\Omega\tau), \qquad (E10)$$

where

$$V(\phi) = \ln(1-\lambda\cos\phi). \qquad (E11)$$

We now write the function $V(\phi)$ in the form of a Fourier series,

$$V(\phi) = \sum_{n=-\infty}^{+\infty} a_n e^{in\phi} \quad \text{with} \quad a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} V(\phi)e^{-in\phi}d\phi. \qquad (E12)$$

The memory function becomes

$$M = -\frac{1}{2}\left(\frac{\gamma}{4\pi r}\right)^2 \int_{-\infty}^{+\infty} d\tau \int_0^{2\pi} d\phi$$

$$\times \sum_{n,m=-\infty}^{+\infty} nma_n a_m e^{i(n+m)\phi}e^{im\Delta\Omega\tau}. \qquad (E13)$$

Carrying out the integrations on ϕ and τ using the integral representation of the delta function

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-i\rho x} d\rho, \tag{E14}$$

we are left with

$$M = -\frac{\gamma^2}{8r^2} \sum_{n,m=-\infty}^{+\infty} nm a_n a_m \delta_{n,-m} \delta(m\Delta\Omega)$$

$$= \frac{\gamma^2}{8r^2} \delta(\Delta\Omega) \sum_{n=-\infty}^{+\infty} |n| a_n^2. \tag{E15}$$

It remains for us to evaluate the series that appears in the last expression of the memory function. Using the identities

$$\int_0^\pi \ln(1-\lambda \cos\phi)\cos(n\phi)d\phi$$

$$= -\frac{\pi}{n}\left(\frac{1}{\lambda} - \sqrt{\frac{1}{\lambda^2} - 1}\right)^n \quad (n>0), \tag{E16}$$

$$\int_0^\pi \ln(1-\lambda \cos\phi)d\phi = \pi \ln\left(\frac{1}{2} + \frac{\sqrt{1-\lambda^2}}{2}\right), \tag{E17}$$

and the definition (E8) of λ, we find that $a_0 < \infty$ and, for $n > 0$,

$$a_n = -\frac{1}{n}\left(\frac{(r_1^2+r^2)-|r_1^2-r^2|}{2rr_1}\right)^n = -\frac{1}{n}\left(\frac{r_<}{r_>}\right)^n, \tag{E18}$$

where $r_>$ ($r_<$) is the biggest (smallest) of r and r_1. Therefore, the value of the series is

$$\sum_{n=-\infty}^{+\infty} |n| a_n^2 = 2\sum_{n=1}^{+\infty} n a_n^2 = 2\sum_{n=1}^{+\infty} \frac{1}{n}\left(\frac{r_<}{r_>}\right)^{2n}$$

$$= -2\ln\left[1-\left(\frac{r_<}{r_>}\right)^2\right]. \tag{E19}$$

The memory function takes the form

$$M = -\frac{\gamma^2}{4r^2} \delta(\Delta\Omega)\ln\left[1-\left(\frac{r_<}{r_>}\right)^2\right] \tag{E20}$$

and the diffusion current in the axisymmetrical case can be written

$$J_r = \frac{N\gamma^2}{4r}\int_0^{+\infty} r_1 dr_1 \delta(\Omega-\Omega_1)\ln\left[1-\left(\frac{r_<}{r_>}\right)^2\right]\left\{\frac{1}{r}P_1\frac{\partial P}{\partial r}\right.$$

$$\left. -\frac{1}{r_1}P\frac{\partial P_1}{\partial r_1}\right\}. \tag{E21}$$

This leads to the kinetic equation (133).

2. Unidirectional flow

If we assume that $P=P(y,t)$, then Eq. (128) simplifies to

$$\frac{\partial P}{\partial t} = -\frac{\partial J_y}{\partial y} \tag{E22}$$

with

$$J_y = -N\int_0^t d\tau \int dx_1 dy_1 V^y(1\rightarrow 0)_t V^y(1\rightarrow 0)_{t-\tau}$$

$$\times\left\{P_1\frac{\partial P}{\partial y} - P\frac{\partial P_1}{\partial y_1}\right\}_{t-\tau}. \tag{E23}$$

Assuming that between t and $t-\tau$ the vortices follow linear trajectories with velocity $\langle\mathbf{V}\rangle=\langle V\rangle(y,t)\mathbf{e}_x$, we have $y(t-\tau)=y$ and $x(t-\tau)=x-\langle V\rangle(y,t)\tau$. Therefore, the function $V^y(1\rightarrow 0)$ at times t and $t-\tau$ takes explicitly the form

$$V^y(1\rightarrow 0)_t = -\frac{\gamma}{2\pi}\frac{x_1-x}{(x_1-x)^2+(y_1-y)^2} \tag{E24}$$

and

$$V^y(1\rightarrow 0)_{t-\tau} = -\frac{\gamma}{2\pi}\frac{x_1-x-\Delta V\tau}{(x_1-x-\Delta V\tau)^2+(y_1-y)^2}, \tag{E25}$$

where we have introduced the notation

$$\Delta V = \langle V\rangle(y_1,t) - \langle V\rangle(y,t). \tag{E26}$$

We also assume that the correlation time is smaller than the time scale over which the vorticity changes appreciably. Then, $P(y(t-\tau),t-\tau)\simeq P(y,t)$ and the diffusion current becomes

$$J_y = -N\int_0^t d\tau \int dx_1 dy_1 V^y(1\rightarrow 0)_t V^y(1\rightarrow 0)_{t-\tau}$$

$$\times\left\{P_1\frac{\partial P}{\partial y} - P\frac{\partial P_1}{\partial y_1}\right\}. \tag{E27}$$

We now need to calculate the memory function

$$M = \int_0^{+\infty} d\tau \int_{-\infty}^{+\infty} dx_1 V^y(1\rightarrow 0)_t V^y(1\rightarrow 0)_{t-\tau}. \tag{E28}$$

Using Eqs. (E24) and (E25), we have explicitly

$$M = \frac{\gamma^2}{4\pi^2}\int_0^{+\infty} d\tau \int_{-\infty}^{+\infty} dX \frac{X}{X^2+Y^2}\frac{X-\Delta V\tau}{(X-\Delta V\tau)^2+Y^2}, \tag{E29}$$

where we have set $X=x_1-x$, $Y=y_1-y$. Equation (E29) can also be written

$$M = \frac{\gamma^2}{4\pi^2} \int_0^{+\infty} d\tau \int_{-\infty}^{+\infty} dX \frac{\partial W}{\partial X}(X,Y) \frac{\partial W}{\partial X}(X-\Delta V\tau, Y)$$

(E30)

with

$$W(X,Y) = \ln\sqrt{X^2+Y^2} = \ln\xi.$$

(E31)

We shall now write the function $W(\xi)$ in the form of a Fourier integral

$$W(\xi) = \frac{1}{(2\pi)^2} \int \hat{W}(\mathbf{k}) e^{-i\mathbf{k}\cdot\xi} d^2\mathbf{k} \quad \text{with}$$

$$\hat{W}(\mathbf{k}) = \int W(\xi) e^{i\mathbf{k}\cdot\xi} d^2\xi.$$

(E32)

Then, the foregoing expression for the memory function becomes

$$M = -\frac{\gamma^2}{(2\pi)^6} \int_0^{+\infty} d\tau \int_{-\infty}^{+\infty} dX \int d^2\mathbf{k} d^2\mathbf{k}' \hat{W}(\mathbf{k}) \hat{W}(\mathbf{k}')$$

$$\times k_x k_x' e^{-i(\mathbf{k}+\mathbf{k}')\cdot\xi} e^{ik_x'\Delta V\tau}.$$

(E33)

Carrying out the integrations over τ and X, we get

$$M = -\frac{\gamma^2}{32\pi^4} \int d^2\mathbf{k} d^2\mathbf{k}' \hat{W}(\mathbf{k}) \hat{W}(\mathbf{k}') k_x k_x'$$

$$\times \delta(k_x + k_x') e^{-i(k_y + k_y')Y} \delta(k_x'\Delta V)$$

(E34)

and, consequently,

$$M = \frac{\gamma^2}{32\pi^4} \delta(\Delta V) \int_{-\infty}^{+\infty} dk_x dk_y dk_y' \hat{W}(k_x, k_y) \hat{W}$$

$$(-k_x, k_y') |k_x| e^{-i(k_y + k_y')Y}.$$

(E35)

Now, the Fourier transform of W can be written explicitly

$$\hat{W}(\mathbf{k}) = 2\pi \int_0^{+\infty} W(\xi) J_0(k\xi) \xi d\xi,$$

(E36)

where use has been made of the well-known identity

$$\int_0^{2\pi} \cos(z\cos\theta) d\theta = 2\pi J_0(z),$$

(E37)

where J_0 is the Bessel function of order zero. It is immediately seen that the Fourier transform of W as defined by Eq. (E31) does not exist. Indeed, the integral (E36) diverges when $\xi \to +\infty$, i.e., at large separations. However, in physical situations the domain never extends to infinity so that, in practice, the integral remains finite. A convenient way to introduce a cutoff at large separations is to make the substitution

$$W(\xi) = \ln\xi \to W(\xi) = -K_0(\xi/L),$$

(E38)

where K_0 is the modified Bessel function of order zero and L is a length scale of the order of the system size. For small separations $K_0(z) \sim -\ln z$ and for large separations $K_0(z) \sim \sqrt{(\pi/2z)} e^{-z}$. This modification amounts to replacing the Poisson equation (2) by an equation of the form

$$-\Delta\psi + \frac{1}{L}\psi = \omega.$$

(E39)

Equation (E39) is precisely what is obtained in geophysics in the "quasigeostrophic approximation." The deformation of the fluid surface introduces a shielding of the interaction between vortices on a length $\sim L$, called the Rossby radius. Obviously, the Rossby radius plays the same role as the Debye length in plasma physics.

With this prescription, we find that

$$\hat{W}(\mathbf{k}) = -\frac{2\pi}{k^2 + k_L^2},$$

(E40)

where $k_L = 1/L$. Substituting in Eq. (E35), we get

$$M = \frac{\gamma^2}{2\pi^2} \delta(\Delta V) \int_{-\infty}^{+\infty} dk_x |k_x| \left(\int_0^{+\infty} \frac{\cos(k_y Y)}{k_L^2 + k_x^2 + k_y^2} dk_y \right)^2.$$

(E41)

The integral on k_y can be carried out easily, leaving the result

$$M = \frac{\gamma^2}{4} \delta(\Delta V) \int_0^{+\infty} dk_x \frac{k_x}{k_L^2 + k_x^2} e^{-2|Y|\sqrt{k_L^2 + k_x^2}}.$$

(E42)

Now, setting $t^2 = 4Y^2(k_x^2 + k_L^2)$, we finally obtain

$$M = \frac{\gamma^2}{4} \delta(\Delta V) E_1\left(\frac{2|Y|}{L}\right),$$

(E43)

where

$$E_1(x) = \int_x^{+\infty} \frac{e^{-t}}{t} dt$$

(E44)

is the exponential integral. For $x \to 0$, we have $E_1(x) = -C - \ln x$, where $C = 0.577\,21\ldots$ is Euler's constant. In conclusion, the diffusion current in the unidirectional case takes the form

$$J_y = -\frac{N\gamma^2}{4} \int_{-\infty}^{+\infty} dy_1 \delta(V_1 - V) E_1\left(\frac{2|y_1 - y|}{L}\right)$$

$$\times \left\{ P_1 \frac{\partial P}{\partial y} - P \frac{\partial P_1}{\partial y_1} \right\}$$

(E45)

and it leads to the kinetic equation (135).

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